1 Harmonic oscillator

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space in a harmonic potential

$$V(\hat{x}) = \frac{\omega^2}{2}\hat{x}^2, \qquad \omega^2 = \frac{k}{m}. \tag{1}$$

We define the *creation* and *annihilation operators* as

$$\hat{a}^\dagger \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \tag{2}$$

$$\hat{a} \coloneqq \frac{\alpha}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{3}$$

$$\alpha \coloneqq \sqrt{\frac{m\omega}{\hbar}}. \tag{4}$$

Proposition 1.1. *Let $\mathcal{H}$ be a Hilbert space in a harmonic potential. Then,*

$$\langle x|\,\hat{a}^\dagger = \frac{\alpha}{\sqrt{2}}\left(x - \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{5}$$

$$\langle x|\,\hat{a} = \frac{\alpha}{\sqrt{2}}\left(x + \frac{1}{\alpha^2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{6}$$

$$\alpha = \frac{m\omega}{\hbar}. \tag{7}$$

Proposition 1.2. *Let $\mathcal{H}$ be a Hilberet space with a harmonic potential. Then,*

$$\hat{x} = \frac{1}{\sqrt{2}\alpha}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\hbar\frac{\alpha}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{8}$$

Proposition 1.3. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{a}, \hat{a}^\dagger$ are not hermitian.*
2. *$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$.*
3. *$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$.*

Definition 1.2. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *number operator* as

$$\hat{N} \coloneqq \hat{a}^\dagger\hat{a}. \tag{9}$$

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{H}$ is hermitian.*
2. *$\left[\hat{N}, \hat{a}\right] = -\hat{a}$, $\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$,*
3. *$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\hat{I}\right)$.*

Proposition 1.5. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{H}$ and $\hat{N}$ have a common basis of eigenvectors, which is countable, and*

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{10}$$

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \qquad \hat{H}\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,, \tag{11}$$

$$n \in \mathbb{N}. \tag{12}$$

Corollary 1.6. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle\,. \tag{13}$$

Proposition 1.7. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, the eigenstates form a non-dgenerate basis.*

Definition 1.3 (Fock states)**.** Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *Fock states* as the states that determine the basis $(|n\rangle)$ and have a well-defined number of excitations.

Definition 1.4. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We call the fundamental Fock state *the vaccum.*

Proposition 1.8. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then, $\hat{a}, \hat{a}^\dagger$ and $\hat{N}$ have the following matrix representation in the basis $(|n\rangle)$.*

$$[\hat{N}]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{14}$$

$$[\hat{a}]_B = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{15}$$

$$[\hat{a}^\dagger]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{16}$$

or in coefficient representation,

$$[\hat{N}]_{ij} = (i-1)\delta_{ij}, \tag{17}$$

$$[\hat{a}]_{ij} = \sqrt{j-1}\delta_{i,j-1}, \tag{18}$$

$$[\hat{a}^\dagger]_{ij} = \sqrt{i-1}\delta_{i-1,j}. \tag{19}$$

Proposition 1.9. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\varphi_0(x) = \langle x|0\rangle = \left(\frac{\beta^2}{\pi}\right)^{1/4}\exp\left(-\frac{\alpha^2x^2}{2}\right), \tag{20}$$

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}x - \frac{1}{\sqrt{2}\beta}\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi_0(x) = \tag{21}$$

$$\frac{1}{\sqrt{2^n n!}}H_n(\beta x)\varphi_0(x). \tag{22}$$

Proposition 1.10. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\hat{\sigma}$ a sequence formed by k $\hat{a}$ and l $\hat{a}^\dagger$. Then,*

$$\langle n|\,\hat{\sigma}\,|n\rangle \leftrightarrow k = l. \tag{23}$$

Proposition 1.11. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\langle \hat{x} \rangle_n = 0, \qquad \langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega}(2n+1), \tag{24}$$

$$\langle \hat{p} \rangle_n = 0, \qquad \langle \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}(2n+1), \tag{25}$$

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1). \tag{26}$$

Proposition 1.12. *Let $\mathcal{H}$ a Hilbert space with a harmonic potential. Then,*

$$\langle T \rangle = \langle V \rangle . \tag{27}$$

Definition 1.5. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define a *coherent state* as a state $|\alpha\rangle \in \mathcal{H}$ such that

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle . \tag{28}$$

Definition 1.6. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displaced state* as the state $|\psi_\alpha\rangle \in \mathcal{H}$ determined by

$$\psi_\alpha(x) = \psi_0(x - x_0). \tag{29}$$

Proposition 1.13. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and a force $F = f$. Then, the fundamental state is a displaced state with $x_0 = f/m\omega^2$.*

Proposition 1.14. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\psi_\alpha\rangle \in \mathcal{H}$ a displaced state with displacement x_0. Then, $|\psi_\alpha\rangle$ is a coherent state with eigenvalue*

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}} x_0. \tag{30}$$

Proposition 1.15. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha \hat{a}^\dagger} |0\rangle . \tag{31}$$

Proposition 1.16. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle$ a coherent state. Then,*

$$\left\langle \hat{N} \right\rangle_\alpha = |\alpha|^2, \qquad p_{|\alpha\rangle}(n) = \mathrm{e}^{-\langle \hat{N} \rangle} \frac{\left\langle \hat{N} \right\rangle^n}{n!}. \tag{32}$$

Theorem 1.17 (Baker-Campbell-Hausdorff formula)**.** *Let $\mathcal{H}$ be a Hilbert space and $\hat{A}, \hat{B} : \mathcal{H} \longrightarrow \mathcal{H}$ two operators such that $\left[\left[\hat{A}, \hat{B}\right], \hat{A}\right], \left[\left[\hat{A}, \hat{B}\right], \hat{B}\right] = 0$. Then,*

$$\exp\left(\hat{A} + \hat{B}\right) = \exp\left(-\frac{1}{2}\left[\hat{A}, \hat{B}\right]\right) \exp\left(\hat{A}\right) \exp\left(\hat{B}\right). \tag{33}$$

Proposition 1.18. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\left[\bar{\alpha}\hat{a}, \alpha\hat{a}^\dagger\right] = |\alpha|^2 \hat{I}, \tag{34}$$

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right) |0\rangle := \hat{\mathcal{D}}(\alpha) |0\rangle . \tag{35}$$

Definition 1.7. Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. We define the *displacement operator* as

$$\hat{\mathcal{D}}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right). \tag{36}$$

Proposition 1.19. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

1. *$\hat{\lceil}(\alpha)$ is unitary.*
2. *$\hat{\mathcal{D}}^\dagger(\alpha) = \hat{\mathcal{D}}(-\alpha)$.*
3. *$\hat{\mathcal{D}}(\alpha)\hat{\mathcal{D}}^\dagger(\alpha) = \hat{I}$.*

Proposition 1.20. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential. Then,*

$$\hat{\mathcal{D}}(\alpha) = \exp\left(-i\frac{x_0\hat{p} - p_0\hat{x}}{\hbar}\right) = \tag{37}$$

$$\exp\left(-\frac{i}{2}\frac{x_0 p_0}{\hbar}\right) \exp\left(i\frac{p_0\hat{x}}{\hbar}\right) \exp\left(-i\frac{x_0\hat{p}}{\hbar}\right), \tag{38}$$

$$x_0 = \sqrt{2} l \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2}\frac{l}{\hbar} \operatorname{Im}\{\alpha\}, \tag{39}$$

$$l = \sqrt{\frac{\hbar}{m\omega}}. \tag{40}$$

Proposition 1.21. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle x|\alpha\rangle = \psi_\alpha(x) = \tag{41}$$

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{ip_0}{2\hbar}(2x - x_0)\right) \exp\left(-\frac{(x-x_0)^2}{4\sigma_x^2}\right), \tag{42}$$

$$\frac{1}{4\sigma_x^2} = \frac{1}{2}\frac{m\omega}{\hbar} \tag{43}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{\alpha\}, \qquad p_0 = \sqrt{2\hbar m\omega} \operatorname{Im}\{\alpha\} \tag{44}$$

Proposition 1.22. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\{|\alpha\rangle\}$ the set of coherent states. Then, they form a* overcomplete basis*, that is, not for all pair of states $|\alpha\rangle, |\alpha'\rangle$ it is satisfied $\langle\alpha'|\alpha\rangle = 0$. Hence,*

$$\hat{I} = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| \, \mathrm{d}^2\alpha \,, \qquad |\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}. \tag{45}$$

Besides,$\langle\alpha|\beta\rangle \to 0$ if and only if $|\alpha - \beta| \gg 1$.

Proposition 1.23. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$|\alpha\rangle(t) = \mathrm{e}^{i\omega t/2} |\alpha(t)\rangle = \mathrm{e}^{i\omega t/2} \left|\alpha_0 \mathrm{e}^{i\omega t}\right\rangle . \tag{46}$$

Proposition 1.24. *Let $\mathcal{H}$ be a Hilbert space witha harmonic potential and $|\alpha\rangle \in \mathcal{H}$ a coherent state. Then,*

$$\langle \hat{x} \rangle = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t), \tag{47}$$

$$\langle \hat{p} \rangle = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t). \tag{48}$$

Definition 1.8. Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a state. We say $|\psi\rangle$ is a *minimum uncertainty state* if and only if

$$\Delta x \Delta p = \frac{\hbar}{2}. \tag{49}$$

Proposition 1.25. *Let $\mathcal{H}$ be a Hilbert state, $|\in\rangle \mathcal{H}$ a state and $|\psi_x\rangle = \hat{\delta x}|\psi\rangle, |\psi_p\rangle = \hat{\delta p}|\psi\rangle$. Then,*

$$\langle\psi_x|\psi_x\rangle \langle\psi_p|\psi_p\rangle \geq |\langle\psi_x|\psi_p\rangle|^2. \tag{50}$$

and the equality only occures when there exists a $\lambda \in \mathbb{C}$ such that $|\psi_p\rangle = \lambda |\psi_x\rangle$.

Proposition 1.26. *Let $\mathcal{H}$ be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ be a state. Then,*

$$\left| \langle\psi|\hat{\delta x}\hat{\delta p}|\psi\rangle\right|^2 \geq \frac{1}{4}\left| \langle\psi|\left[\hat{\delta x}, \hat{\delta p}\right]|\psi\rangle\right|^2, \tag{51}$$

and the equality only occures when $\{\hat{\delta x}, \hat{\delta p}\} = 0$.

Proposition 1.27. *Let $\mathcal{H}$ be a Hilbert space and $|\in\rangle \mathcal{H}$ a minimum uncertainty state. Then,*

$$\langle x|\psi\rangle = \psi(x) = \tag{52}$$

$$C \exp\left[-\frac{|\lambda|}{2}(x - \langle x\rangle)^2\right] \exp\left[\frac{ix\langle p\rangle}{\hbar}\right], \tag{53}$$

for some $\lambda \in \mathbb{C}$ and with variance $\Delta x^2 = \hbar/2|\lambda|$.

Proposition 1.28. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $\hat{b} = \hat{a} - \alpha\hat{I}$. Then,*

$$|\alpha\rangle = |0_\alpha\rangle, \qquad \hat{b}\,|0_\alpha\rangle = 0, \qquad \hat{N}_b = \hat{b}^\dagger\hat{b}, \tag{54}$$

$$\left[\hat{b}, \hat{b}^\dagger\right] = \hat{I}, \qquad \hat{N}_b\,|n\rangle_b = n\,|n\rangle_b\,, \tag{55}$$

$$\hat{b}\,|n\rangle_b = \sqrt{n+1}\,|n+1\rangle_b\,. \tag{56}$$

Proposition 1.29. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential, $\alpha = \sqrt{\frac{m\omega}{2\hbar}}x_0$ and $\hat{H} = \hbar\omega\left(\frac{1}{2} + \hat{N}_b\right)$. Then,*

$$\hat{H}' = \frac{\hat{p}^2}{wm} + \frac{m\omega^2}{2}(\hat{x} - x_0)^2 - \frac{m\omega^2}{2}x_0^2. \tag{57}$$

Proposition 1.30 (Bogoliubov's transformation)**.** *Let $\mathcal{H}$ be a Hilbert space with a variant harmonic potential*

$$V(\hat{x}) = \begin{cases} \dfrac{m_a\omega_a^2}{2}\hat{x}^2 & t < 0, \\ \dfrac{m_b\omega_b^2}{2}\hat{x}^2 & t \geq 0 \end{cases}. \tag{58}$$

Then,

$$\begin{cases} \hat{a} = \hat{b}\cosh\gamma + \hat{b}^\dagger\sinh\gamma, \\ \hat{a}^\dagger = \hat{b}\sinh\gamma + \hat{b}^\dagger\cosh\gamma \end{cases}, \tag{59}$$

$$\begin{cases} \hat{b} = \hat{a}\cosh\gamma - \hat{a}^\dagger\sinh\gamma, \\ \hat{b}^\dagger = -\hat{a}\sinh\gamma + \hat{a}^\dagger\cosh\gamma \end{cases}. \tag{60}$$

Proposition 1.31. *Let $\mathcal{H}$ be a Hilbert space with a variant harmonic potential. Then,*

$$|0_\gamma\rangle = |0\rangle_a = \frac{1}{\sqrt{\cosh\gamma}}\exp\left[-\frac{1}{2}\tanh\gamma(\hat{b}^\dagger)^2\right]|0\rangle_b\,, \tag{61}$$

$$\ln\sqrt{\frac{m_a\omega_a}{m_b\omega_b}}. \tag{62}$$

Proposition 1.32. *Let $\mathcal{H}$ be a Hilbert space with a variant harmonic potential. Then,*

$$|0_\gamma\rangle = \hat{S}(\gamma)\,|0\rangle_b = \exp\left[-\frac{\gamma}{2}(\hat{b}^{\dagger 2} - \hat{b}^2)\right]|0\rangle_b\,. \tag{63}$$

We call $\hat{S}(\gamma)$ the squeezing operator*.*

Proposition 1.33. *Let $\mathcal{H}$ be a Hilbert space witha avariant harmonic potential. Then,*

1. *If $\gamma \to \infty$, then $\Delta x \to 0$ and $|0_\gamma\rangle \to |x\rangle$.*
2. *If $\gamma \to -\infty$, then $\Delta p \to 0$ and $|0_\gamma\rangle \to |p\rangle$.*

Proposition 1.34. *Let $\mathcal{H}$ be a Hilbert space with a harmonic potential and $|\psi\rangle \in \mathcal{H}$ a state. Then,*

1. *If $|\psi\rangle$ is the vacuum state, Δp, Δx are constant.*
2. *If $|\psi\rangle$ is an squeezed state, Δp, Δx vary.*

Proposition 1.35. *Let $\mathcal{H}$ be a Hilbert space, $\hat{a}, \hat{a}^\dagger$ ladder operators and $f(\hat{a}, \hat{a}^\dagger), f^\dagger(\hat{a}, \hat{a}^\dagger)$ other ladder operators. Then, their general form is*

$$f(\hat{a}, \hat{a}^\dagger) = \alpha\hat{I} + z_1\hat{a} + z_2\hat{a}^\dagger, \tag{64}$$

$$\alpha, z_1, z_2 \in \mathbb{C},\ |z_1|^2 - |z_2|^2 = 1. \tag{65}$$

Proposition 1.36. *Let $\mathcal{H}$ be a Hilbert space. Then, squeezed states are the vacuum states of the operator*

$$\hat{a}_\gamma = \cosh\gamma\hat{a} + \sinh\gamma\hat{a}^\dagger. \tag{66}$$

Proposition 1.37. *Let $\mathcal{H}$ be a Hilbert space. Then, coherent states are the vacuum states of the operator*

$$\hat{a}_\alpha = \hat{a} - \alpha\hat{I}. \tag{67}$$

Proposition 1.38. *Let $\mathcal{H}$ be a Hilbert space. Then, the time dependent coherent states $|\alpha\rangle(t)$ are the coherent states of the operator*

$$\hat{a}_t = \mathrm{e}^{-i\omega t}\hat{a}. \tag{68}$$

2 Angular momentum

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space. We define the *angular momentum operator* on $\mathcal{H}$ as the generator of rotations

$$\mathcal{D}_{\mathbf{n}}(\theta) = \exp\biggl(-\frac{i\theta}{\hbar}\langle\mathbf{n}, \mathbf{J}\rangle_I\biggr). \tag{69}$$

Proposition 2.1. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\sum_k \epsilon_{ijk}\hat{J}_k. \tag{70}$$

Proposition 2.2. *Let $\mathcal{H}$ be a Hilbert space. Then, the angular momentum operator is hermitian, that is, $\hat{J}_i^\dagger = \hat{J}_i\ \forall i$.*

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space. We define the *squared angular momentum operator* as

$$\hat{J}^2 \coloneqq \langle \mathbf{J}, \mathbf{J} \rangle_I. \tag{71}$$

Definition 2.3.

$$\hat{J}_{\mathbf{n}} \coloneqq \langle \mathbf{n}, \mathbf{J} \rangle_I. \tag{72}$$

Proposition 2.3. *Let $\mathcal{H}$ be a Hilbert space. Then, $\hat{J}^2$ is unvariant under rotations, that is,*

$$\left[\hat{J}^2, \hat{J}_{\mathbf{n}}\right] = 0, \ \forall \mathbf{n}. \tag{73}$$

Proposition 2.4. *Let $\mathcal{H}$ be a Hilbert space and $(|\beta, m\rangle)$ a common eigenbasis of $\hat{J}^2$ and $\hat{J}_z$. Then,*

$$\beta \geq m^2. \tag{74}$$

Definition 2.4. Let $\mathcal{H}$ be a Hilbert space and $\hat{J}_i$ the angular momentum opperators. We define their *ladder operators* as

$$\hat{J}_+ \coloneqq \hat{J}_x + i\hat{J}_y, \qquad \hat{J}_- \coloneqq \hat{J}_x - i\hat{J}_y = \hat{J}_+^\dagger. \tag{75}$$

Proposition 2.5. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\begin{cases} \hat{J}_x = \dfrac{1}{2}\hat{J}_+ + \dfrac{1}{2}\hat{J}_- \\ \hat{J}_y = -\dfrac{i}{2}\hat{J}_+ + \dfrac{i}{2}\hat{J}_- \end{cases} \tag{76}$$

Proposition 2.6. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm, \tag{77}$$

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z, \tag{78}$$

$$\left[\hat{J}^2, \hat{J}_\pm\right] = 0. \tag{79}$$

Proposition 2.7. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar J_z, \tag{80}$$

$$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar J_z. \tag{81}$$

Proposition 2.8. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\hat{J}_\pm |j, m\rangle \propto |j, m \pm 1\rangle, \tag{82}$$

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle, \tag{83}$$

$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle. \tag{84}$$

Proposition 2.9. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$\hat{J}_+ |j, m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle, \tag{85}$$

$$\hat{J}_- |j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle. \tag{86}$$

Definition 2.5. Matrix representation of $\hat{J}_z$

$$[\hat{J}_z] = \delta_{jj'}\delta_{mm'}\hbar m. \tag{87}$$

Corollary 2.10. *Metrix representation of $\hat{J}_z$ for $j = 0, 1/2, 1, 3/2$*

$$[\hat{J}_z^0] = \begin{pmatrix} 0 \end{pmatrix}, \tag{88}$$

$$[\hat{J}_z^{1/2}] = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \tag{89}$$

$$[\hat{J}_z^1] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{90}$$

$$[\hat{J}_z^{3/2}] = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}. \tag{91}$$

Proposition 2.11. *Matrix representation of $\hat{J}_\pm$*

$$[\hat{J}_\pm] = \hbar\sqrt{j(j+1) - m(m \pm 1)}\delta_{j'j}\delta_{m'm\pm 1}. \tag{92}$$

Corollary 2.12. *Metrix representation of $\hat{J}_\pm$ for $j = 0, 1/2, 1, 3/2$*

$$[\hat{J}_+^0] = \begin{pmatrix} 0 \end{pmatrix}, \tag{93}$$

$$[\hat{J}_+^{1/2}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{94}$$

$$[\hat{J}_+^1] = \begin{pmatrix} 0 & \sqrt{2} & 1 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{95}$$

$$[\hat{J}_+^{3/2}] = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{96}$$

$$[(\hat{J}_+^0)^2] = \begin{pmatrix} 0 \end{pmatrix}, \tag{97}$$

$$[(\hat{J}_+^{1/2})^2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{98}$$

$$[(\hat{J}_+^1)^2] = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$[(\hat{J}_+^{3/2})^2] = \begin{pmatrix} 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{100}$$

Proposition 2.13. *Matrix representation of $\hat{J}^2$.*

$$[\hat{J}^2] = \hbar^2 j(j+1)\delta_{jj'}\delta_{mm'}. \tag{101}$$

Definition 2.6. Let $\mathcal{H}$ be a Hilbert space. We define the *orbital angular momentum operator* as

$$\mathbf{L} \coloneqq \mathbf{r} \times \mathbf{p} = -i\hbar\mathbf{r} \times \nabla. \tag{102}$$

Proposition 2.14. *Let $\mathcal{H}$ be a Hilbert space. Then,*

1. $\left[\hat{L}_i, \hat{L}_j\right] = \sum_k \epsilon_{ijk}\hat{L}_k$
2. $\left[\hat{L}^2, L_i\right] = 0 \ \forall i, \left[\hat{L}^2, \mathbf{L}\right] = \mathbf{0}.$

Proposition 2.15. *Let $\mathcal{H}$ be a Hilbert space. Then,*

1. *Cartensian basis representation*

$$\langle \mathbf{r}| \mathbf{L} = -i\hbar(y\partial_z - z\partial_y)\mathbf{e}_x . i\hbar(z\partial_x - x\partial_z)\mathbf{e}_y - i\hbar(x\partial_y - y\partial_x)\mathbf{e}_z. \tag{103}$$

2. Spherical basis representation

$$\langle \mathbf{r}| \, \mathbf{L} = -i\hbar \frac{\partial}{\partial \theta}\mathbf{e}_\varphi + i\hbar \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\mathbf{e}_\theta. \tag{104}$$

3. Sherical parameters representation

$$\langle \mathbf{r}| \, \hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \tag{105}$$

$$\langle \mathbf{r}| \, \hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}. \tag{106}$$

Proposition 2.16. *Let $\mathcal{H}$ be a Hilbert space. Then,*

$$L_z Y_l^m(\theta,\varphi) = \hbar m Y_l^m(\theta,\varphi) \tag{107}$$

$$L^2 Y_l^m(\theta,\varphi) = \hbar^2 l(l+1) Y_l^m(\theta,\varphi), \tag{108}$$

with

$$Y_l^m(\theta,\varphi) = \tag{109}$$

$$(-1)^{\frac{m+|m|}{2}}\left[\frac{(2l+1)(l-|m|)!}{4\pi(l-|m|)!}\right]^{1/2} \mathrm{e}^{im\varphi} P_l^{|m|}(\cos\theta). \tag{110}$$

Definition 2.7. Let $\mathcal{H}$ be a Hilbert space and $\mathbf{L}$ the orbital angular momentum operator. We define its *ladder operators* as

$$L_+ \coloneqq L_x + iL_y = \hbar \mathrm{e}^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right], \tag{111}$$

$$L_- \coloneqq L_x - iL_y = \hbar \mathrm{e}^{i\varphi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right] \tag{112}$$