1 Mathematical formulism of quantum mechanics

Definition 1.1. We define a *Hilbert space* $\mathcal{H}$as a vector space over the field $\mathbb{C}$ where there is an inner product $\langle , \rangle$ that has the properties of

1. Linearity: $\langle\phi|\left(a\left|\psi_1\right\rangle+b\left|\psi_2\right\rangle\right) = a\left\langle\phi|\psi_1\right\rangle + b\left\langle\phi|\psi_2\right\rangle$,
2. Positivity: $\langle\psi|\psi\rangle > 0, \forall\psi \neq 0$,
3. Hermiticity: $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$,

and such that is complete in the norm $\||\psi\rangle\| := \sqrt{\langle\psi|\psi\rangle}$.

Definition 1.2. Let $\mathcal{H}$be a Hilbert space. We define an *orthonormal basis* $\mathcal{B} = (|e_i\rangle)$ as a collection of vectors that satisfy the following conditions

1. $\forall|\psi\rangle \in \mathcal{H}\ \exists!(\alpha_1, \ldots, \alpha_n)$ such tat $|\psi\rangle = \sum_{i=1}^n \alpha_i |e_i\rangle$,
2. $\forall i, j \leq n\ \langle e_i|e_j\rangle = \delta_{ij}$.

Proposition 1.1. *Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$|\psi\rangle = \sum_{i=1}^n \alpha_i |e_i\rangle, \text{ with } \alpha_i = \langle e_i|\psi\rangle. \tag{1}$$

Definition 1.3. Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. We define the *representation of $|\psi\rangle$ in the basis $\mathcal{B}$* as

$$|\psi\rangle_{\mathcal{B}} := \begin{pmatrix} \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}_{(|e_i\rangle)}. \tag{2}$$

Definition 1.4. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a vector. We define a *bra* as

$$\langle\psi| := |\psi\rangle^\dagger \in \mathcal{H}. \tag{3}$$

Proposition 1.2. *Let $\mathcal{H}$be a Hilbert space, $|\psi\rangle \in \mathcal{H}$ a vector and $\mathcal{B} = (|e_i\rangle)$ a basis. If $|\psi\rangle = \sum \alpha_i i |e_i\rangle$, then*

$$\langle\psi| = \sum_{i=1}^n \overline{\alpha_i} \langle e_i|. \tag{4}$$

Definition 1.5. Let $\mathcal{H}$be a Hilbert space. we define the *inner product* as the function $f_{in} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ that acts as follows

$$f_{in}(\langle\psi|, \langle\phi|) := \langle\psi|\phi\rangle = |\psi\rangle^\dagger |\phi\rangle = \tag{5}$$

$$\begin{pmatrix} \overline{\alpha_1} & \ldots & \overline{\alpha_n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \sum_{i=1}^n \overline{\alpha_i}\beta_i. \tag{6}$$

Proposition 1.3. *The inner product satisfies the conditions of hermitic product for a Hilbert space.*

Definition 1.6. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ two vectors. We say they are *orthogonal* if and only if $\langle\psi|\phi\rangle = 0$.

Definition 1.7. Let $\mathcal{H}$be a Hilbert space and $|\psi\rangle \in \mathcal{H}$ a vector. We say it is normalized if and only if $\langle\psi|\psi\rangle = 1$.

Definition 1.8. Let $\mathcal{H}$ be a Hilbert space. We dfien the *exterior product* as the function $f_{ext} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}_{2\times 2}$ that acts as follows

$$f_{ext}(|\psi\rangle, |\phi\rangle) := |\psi\rangle\langle\phi| = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \begin{pmatrix} \overline{\beta_1} & \ldots & \overline{\beta_n} \end{pmatrix} = \tag{7}$$

$$\begin{pmatrix} \alpha_1\overline{\beta_1} & \ldots & \alpha_1\overline{\beta_n} \\ \vdots & \ddots & \vdots \\ \alpha_n\overline{\beta_1} & \ldots & \alpha_n\overline{\beta_n} \end{pmatrix}. \tag{8}$$

Definition 1.9. Let $\mathcal{H}$ be a Hilbert space. we define a *linear operator* as an application $A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ such that if $a, b \in \mathbb{C}$ and $\psi_1, \psi_2 \in \mathcal{H}$, then

$$A\left(a\left|\psi_1\right\rangle + b\left|b_2\right\rangle\right) = aA\left(\left|\psi_1\right\rangle\right) + bA\left(\left|\psi_2\right\rangle\right). \tag{9}$$

We denote the matrix form of A as $\hat{A}$.

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space, A an operator and $B = (|e_i\rangle)$ a basis. If $|\psi\rangle = \sum \alpha_i |e_i\rangle$, then $\hat{A}|\psi\rangle$ is uniquely determined by the elements $|f_i\rangle = |f(e_i)\rangle$ as follows*

$$A(|\psi\rangle) = \sum_{i=1}^n \alpha_i |f_i\rangle. \tag{10}$$

Proposition 1.5. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A} = \sum_{i=1}^n |f_i\rangle\langle e_i|. \tag{11}$$

Proposition 1.6. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A} = \sum_{i=1}^n A_{ij} |e_i\rangle\langle e_j|, \text{ with } A_{ij} = f_i^{(j)} = \langle e_i| \hat{A} |e_j\rangle. \tag{12}$$

Proposition 1.7. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $\mathcal{B} = (|e_i\rangle)$ a basis. Then,*

$$\hat{A}|e_k\rangle = \sum_{l=1}^n \hat{a}_{lk} |e_l\rangle. \tag{13}$$

Proposition 1.8. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator, $\mathcal{B} = (|e_i\rangle)$ a basis and $|\psi\rangle \in \mathcal{H}$ a vector. If $|\psi\rangle = \sum \alpha_i |e_i\rangle$ and $|\phi\rangle = \hat{A}|\psi\rangle = \sum \beta_i |e_i\rangle$, then*

$$\beta_j = \langle e_j|\phi\rangle = \sum_{k=1}^j \hat{A}_{jk}\alpha_k. \tag{14}$$

Definition 1.10. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operatorr. We define the *action by the right* of as the following way.

$$\left(\langle\phi| \hat{A} \right) |\psi\rangle \coloneqq \langle\phi| \left(\hat{A} |\psi\rangle \right), \ \forall |\psi\rangle, |\phi\rangle \in \mathcal{H} \tag{15}$$

Definition 1.11. Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator that transforms $|\psi\rangle$ in $|\phi\rangle = \hat{A}|\psi\rangle$. We define the *adjoint operator* $\hat{A^\dagger}$ as the operator that transforms $\langle\psi|$ to $\langle\phi| = \langle\psi| \hat{A}^\dagger$.

Proposition 1.9. *Let $\mathcal{H}$ be a Hilbert space, $A, B : \mathcal{H} \longrightarrow \mathcal{H}$ two operators and $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ two vectors. Then,*

1. $|\psi\rangle^\dagger = \langle\psi|$,
2. $(|\phi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\phi|$,
3. $(\lambda\hat{A})^\dagger = \overline{\lambda}\hat{A}^\dagger$,
4. $\left(\hat{A}|\psi\rangle\right)^\dagger = \langle\psi| \hat{A}^\dagger$,
5. $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

Proposition 1.10. *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator. Then,*

$$\hat{A^\dagger} = \hat{A}^\dagger, \ [A^\dagger]_{ij} = [A]^*_{ij}. \tag{16}$$

Definition 1.12. Let $\mathcal{H}$ be a Hilbert space. We define the *identity operator* $I : \mathcal{H} \longrightarrow \mathcal{H}$ as the operator that satisies

$$\hat{I} |\psi\rangle = |\psi\rangle, \ \forall |\psi\rangle \in \mathcal{H}. \tag{17}$$

Proposition 1.11. *Let $\mathcal{H}$ be a Hilbert space and $B = (|e_i\rangle)$ an orthonormal basis. Then,*

$$\hat{I} = \sum_{i=1}^{n} |e_i\rangle\langle e_i|. \tag{18}$$

Proposition 1.12. *Let $\mathcal{H}$ be a Hilbert space. Then, $\hat{I}$ is independent of the orthonormal basis.*

Definition 1.13. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator. We define the *trace of* A as the trac of its matrix $\hat{A}$, that is,

$$\operatorname{tr} A \coloneqq \operatorname{tr} \hat{A} = \sum_{I01}^{n} \langle e_i| \hat{A} |e_i\rangle. \tag{19}$$

Definition 1.14. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator. Then, $\operatorname{tr} A$ is independent of the orthonormal basis.

Proposition 1.13. *Let $\mathcal{H}$ be a Hilbert space and $A, B : \mathcal{H} \longrightarrow \mathcal{H}$ two operators. Then,*

$$\operatorname{tr}\left(\hat{A}\hat{B}\right) = \operatorname{tr}\left(\hat{B}\hat{A}\right). \tag{20}$$

Corollary 1.14. *Let $\mathcal{H}$ be a Hilbert space and $A, B, C : \mathcal{H} \longrightarrow \mathcal{H}$ three operators. Then,*

$$\operatorname{tr}\left(\hat{A}\hat{B}\hat{C}\right) = \operatorname{tr}\left(\hat{C}\hat{A}\hat{B}\right) = \operatorname{tr}\left(\hat{B}\hat{C}\hat{A}\right). \tag{21}$$

Proposition 1.15. *Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator and $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ two vectors. Then,*

$$\operatorname{tr}\left(\hat{A} |\psi\rangle\langle\phi|\right) = \langle\phi| \hat{A} |\psi\rangle. \tag{22}$$

Definition 1.15. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator. We say A is *hermitian* or *auto-adjoint* if and only if $A = A^\dagger$.

Proposition 1.16. *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an hermitian operator. Then, $\hat{A}$ is diagonalizable and there exists an orthonormal basis $\mathcal{B} = (|e_i\rangle)$ such that*

$$\hat{A} = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|, \tag{23}$$

where λ_is are the eigenvalues and $|e_i\rangle$s the eigenvectors of $\hat{A}$.

Theorem 1.17. *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ a hermitian operator. Then,*

1. *The eigenvalues λ_i are all real.*
2. *The eigenvectors with different eigenvalues are orthogonal, that is, $\langle e_i|e_j\rangle = 0$.*

Definition 1.16. Let $\mathcal{H}$ be a Hilbert space, $A : \mathcal{H} \longrightarrow \mathcal{H}$ a hermitian operator and $|e_1\rangle, \ldots, |e_r\rangle$ the eigenvectors of $\hat{A}$. We say $|e_i\rangle$, $|e_j\rangle$ are *degenerate* if and only if they have the same eigenvalue. In this case, we define the *degree of the egeneration of an eigenvalue* λ as $\dim E_\lambda$.

Definition 1.17. Let $\mathcal{H}$ be a Hilbert space, $G \subseteq \mathcal{H}$ a subspace of dimensioni m and $\mathcal{B} = (|g_i\rangle)$ an orthonormal basis of G. We define the *projector to the subspace* G as the operator $P : \mathcal{H} \longrightarrow G$ determined by

$$\hat{P} = \sum_{i=1}^{r} |g_i\rangle\langle g_i|. \tag{24}$$

Proposition 1.18. *Let $\mathcal{H}$ be a Hilbert space, $V_1, \ldots, V_m \subseteq \mathcal{H}$ subspaces and $P_1 : \mathcal{H} \longrightarrow G_1, \ldots, P_m :\longrightarrow G_m$ the projectors to these spaces. Then,*

$$\hat{P}_i\hat{P}_j = \delta_{ij}\hat{P}_j. \tag{25}$$

Proposition 1.19. *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an hermitian operator. Let $\lambda_1, \ldots, \lambda$ be the different eigenvalues of $\hat{A}$ with their subspaces $V_1, \ldots, V_m$ of diensions $d_1, \ldots, d_m$ and $P_1 : \mathcal{H} \longrightarrow V_1, \ldots, P_m : \mathcal{H} \longrightarrow V_m$ the projectors to these subspaces. Then,*

$$\hat{A} = \sum_{i=1}^{m} \lambda_i\hat{P}_i. \tag{26}$$

We call this expression the spectral decompossition.