# 1 The Rn space

**Definition 1.1.** Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ *is convergent to the real number* $L$ (or *has limit* $L$) if it satisfies the following condition.

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |a_n - L| < \varepsilon, \forall n > n_0 \quad (1)$$

And we will write $\{a_n\} \to L$ or $\lim \{a_n\} = L$

**Definition 1.2.** Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ *is a Cauchy sequence* if it satisfies the following condition.

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid |a_m - a_n| < \varepsilon, \forall m, n > n_0 \quad (2)$$

**Definition 1.3.** We define a *ball* $B$ *of center* $a$ *and radius* $r$ as a subset of $\mathbb{R}$ determined as follows.

$$B(a, r) \coloneqq \{x \in \mathbb{R} \mid |x - a| < r\} \quad (3)$$

**Definition 1.4.** We define a *punctured ball* $B^*$ *of center* $a$ *and radius* $r$ as a neighborhood of center $a$ and radius $r$ that does not contain $a$. It can also be expressed as follows.

$$B^*(a, r) \coloneqq \{x \in \mathbb{R} \mid 0 < |x - a| < r\} \quad (4)$$

**Definition 1.5.** Let $A \subseteq \mathbb{R}$ be a set. We say that $a$ is an *interior point of* $A$ if there is a neighborhood $\mathfrak{N}(a, r) \subset A$.

**Definition 1.6.** Let $A \subseteq \mathbb{R}$ be a set. We say that $a$ is an *exterior point of* $A$ if it is an interior point of $A^c$.

**Definition 1.7.** Let $A \subseteq \mathbb{R}$ be a set. We say that $a$ is a *boundary point of* $A$ if it is not interior or exterior.

**Definition 1.8.** Let $A \subseteq \mathbb{R}$ be a set. We say that $a$ is an *accumulation point of* $A$ if every neighborhood with center $a$ contains points of $A$ different to $a$.

**Definition 1.9.** Let $A \subseteq \mathbb{R}$ be a set. We say $A$ *is an open set* if all its points are interior points.

**Definition 1.10.** Let $A \subseteq \mathbb{R}$ be a set. We say $A$ *is a closed set* if it contains all its accumulation points (or, what is equivalent, if $A^c$ is open).

**Definition 1.11.** Let $A \subseteq \mathbb{R}$ be a set. We say $A$ *is a bounded set* if it is contained by some neighborhood of center 0.

**Definition 1.12.** Let $A \subseteq \mathbb{R}$ be a set. We say $A$ *is a compact set* if every sequence of elements of $A$ has some partial sequence convergent inside $A$ (or, what is equivalent, if it is closed and bounded).

**Definition 1.13.** A *metric space* $(\mathbb{M}, d)$ is a set $\mathbb{M}$ with a *metric* or *distance function* $d$ of the form

$$\begin{aligned} d : \mathbb{M} \times \mathbb{M} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto d(x, y) \end{aligned} \quad (5)$$

That satisfies the following properties:

1. For any element $x \in \mathbb{M}$, we have $d(x, x) = 0$
2. Positivity: for any distinct $x, y \in \mathbb{M}$, we have $d(x, y) > 0$
3. Symmetry: for any $x, y \in \mathbb{M}$, we have $d(x, y) = d(y, x)$
4. Triangle inequality: for any $x, y, z \in \mathbb{M}$, we have $d(x, z) \leq d(x, y) + d(y, z)$

**Definition 1.14.** Let $A$ be a set. A *sequence in* $A$ is an application from natural numbers to $A$.

$$\begin{aligned} \mathbb{N} &\longrightarrow A \\ k &\longmapsto a_k \end{aligned} \quad (6)$$

We denote it by $\{a_1, a_2, \dots\}$ or, more shortly, $\{a_k\}$.

**Definition 1.15.** Let $\{a_k\}$ be a sequence of points of o a metric space $(\mathbb{M}, d)$. We say *the sequence* $\{a_k\}$ *is bounded* is there exist a point $a$ and a positive real number $r$ such that $d(a_k, a) < r$.

**Definition 1.16.** Let $\{a_k\}$ be a sequence of points of o a metric space $(\mathbb{M}, d)$. We say that $\{a_k\}$ *is convergent to the point* $L \in (\mathbb{M}, d)$ (or *has limit* $L$) *with respect to the metric* $d$ if it satisfies the following condition.

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid d(a_k, L) < \varepsilon, \forall n > n_0 \quad (7)$$

In that case, we will write $\{a_k\} \to L$ or $\lim_{k\to\infty} \{a_k\} = L$. As before [], this is equivalent to say it is convergent to $L$ if equivalent

**Proposition 1.1.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $\{a_k\}$ *a sequence in* $\mathbb{M}$ *that is convergent to a point* $L$ *with respect to the metric* $d$*. Then,* $L$ *is unique.*

**Proposition 1.2.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $\{a_k\}$ *a sequence in* $\mathbb{M}$ *that is convergent to a point* $L$ *with respect to the metric* $d$*. Then,* $\{a_k\}$ *is bounded.*

**Definition 1.17.** Let $\{a_k\}$ be a sequence of points of $(\mathbb{M}, d)$. We say that $\{a_k\}$ *is a Cauchy sequence with respect to the metric* $d$ if it satisfies the following condition.

$$\forall \varepsilon \in \mathbb{R}^+ \exists n_0 \in \mathbb{N} \mid d(x_k, x_l) < \varepsilon, \forall l, k > n_0 \quad (8)$$

**Proposition 1.3.** *Let* $(\mathbb{M}, d)$ *a metric space and* $\{a_k\}$ *a convergent sequence in* $(\mathbb{M}, d)$*. Then,* $\{a_k\}$ *is a Cauchy sequence.*

**Definition 1.18.** Let $(\mathbb{M}, d)$ a metric space. We say $(\mathbb{M}, d)$ is *complete* if every Cauchy sequence in $(\mathbb{M}, d)$ is a convergent sequence in $(\mathbb{M}, d)$.

**Definition 1.19.** Let $(\mathbb{M}, d)$ be a metric space, $a$ a point in $\mathbb{M}$, and $r$ a positive real number. We define a *metric ball* $B_{(\mathbb{M},d)}$ *of center* $a$ *and radius* $r$ *in the metric* $d$ as the set

$$B_{(\mathbb{M},d)}(a, r) \coloneqq \{x \in \mathbb{M} \mid d(x, a) < r\}. \quad (9)$$

If the metric space $(\mathbb{M}, d)$ is clear, we just write $B(a, r)$

**Definition 1.20.** Let $(\mathbb{M}, d)$ be a metric space, $a$ a point in $\mathbb{M}$, and $r$ a positive real number. We define a *closed metric ball* $B^{\wr}_{(\mathbb{M},d)}$ *of center* $a$ *and radius* $r$ *in the metric* $d$ as the set

$$B^{\wr}_{(\mathbb{M},d)}(a, r) \coloneqq \{x \in \mathbb{M} \mid d(x, a) \leq r\}. \quad (10)$$

**Definition 1.21.** Let $(\mathbb{M}, d)$ be a metric space, $a$ a point in $\mathbb{M}$, and $r$ a positive real number. We define a *punctured metric ball* $B^*_{(\mathbb{M},d)}$ *of center* $a$ *and radius* $r$ *in the metric* $d$ as the set

$$B^*_{(\mathbb{M},d)}(a, r) := \{x \in \mathbb{M} \,|\, 0 < d(x, a) < r\}. \quad (11)$$

**Definition 1.22.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *interior point of* $A$ if there is a ball $B_{(\mathbb{M},d)}(a, r) \subset A$.

**Definition 1.23.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *exterior point of* $A$ if there is a ball such that $B_{(\mathbb{M},d)}(a, r) \cup A = \emptyset$.

**Definition 1.24.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is a *boundary point of* $A$ if it is not interior or exterior or, which is equivalent, if every ball $B_{(\mathbb{M},d)}(a, r)$ contains elements of $A$ and $A^c$.

**Definition 1.25.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$ and $a$ a point in $\mathbb{M}$. We say that $a$ is an *accumulation point of* $A$ if every ball with center $a$ contains points of $A$ different to $a$. In other words, every punctured ball satisfies $B^*_{(\mathbb{M},d)}(a, r) \cup A \neq \emptyset$.

**Definition 1.26.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the interior of* $A$ as the set of all interior points of $A$, and we denote it by $\text{int}(A)$.

**Definition 1.27.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the exterior of* $A$ as the set of all exterior points of $A$, and we denote it by $\text{ext}(A)$.

**Definition 1.28.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the boundary of* $A$ as the set of all boundary points of $A$, and we denote it by $\partial A$.

**Definition 1.29.** Let $A$ be a subset of a metric space $(\mathbb{M}, d)$. We define *the closure of* $A$ as the set of all accumulation points of $A$, and we denote it by $\bar{A}$.

**Proposition 1.4.** *Let* $A$ *be a subset of a metric space* $(\mathbb{M}, d)$ *and* $a$ *an exterior point of* $A$*. Then, it is an interior point of* $A^c$*.*

**Proposition 1.5.** *Let* $A$ *be a subset of a metric space* $(\mathbb{M}, d)$ *and* $a$ *an exterior point of* $A$*. Then, the following statements are equivalent* **?**.

1. *$a$ is an accumulation point of $A$.*
2. *$a$ is either an interior point or a boundary point of $A$.*
3. *There exists a sequence $\{a_k\}$ in $A$ which converges to $a$ with respect to the metric $a$.*

**Corollary 1.6.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $\bar{A} = int(A) \cap \partial A = \mathbb{M} \setminus ext(A)$*.*

**Definition 1.30.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *an open set* if it contains none of its boundary points, that is, if $\partial A \cap A = \emptyset$.

**Definition 1.31.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a closed set* if it contains all its boundary points, that is, if $\partial A \subseteq A$.

**Definition 1.32.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a bounded set* if there exist a point $a \in \mathbb{M}$ and a positive real number $r$ such that the ball $B_{(\mathbb{M},d)}(a, r)$ contains $A$.

**Definition 1.33.** Let $(\mathbb{M}, d)$ be a metric space and $A$ a subset of $\mathbb{M}$. We say $A$ is *a compact set* if it a bounded and closed set.

**Proposition 1.7.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *an open subset of* $\mathbb{M}$*. Then, all its points are interior points of* $A$*.*

**Proposition 1.8.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $A$ *is open if and only if* $A = int(E)$*. In other words,* $A$ *is open if and only if for every* $a \in A$*, there exists an* $r \in \mathbb{R}^+$ *such that* $B(a, r) \subseteq A$*.*

**Proposition 1.9.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $A$ *is closed if and only if* $A$ *contains all its accumulation points, that is,* $\hat{A} = A$*. In other words,* $A$ *is closed if and only if for every convergent sequence* $\{a_k\}$ *in* $A$*, the limit* $\lim_{k\to\infty} \{a_k\}$ *of that sequence also lies in* $A$*.*

**Proposition 1.10.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then, for any* $a \in A$ *and* $r \in \mathbb{R}^+$ *the ball* $B_{(\mathbb{M},d)}(a, r)$ *is an open set and the closed ball* $B^{\wr}_{(\mathbb{M},d)}(a, r)$ *is a closed set.*

**Proposition 1.11.** *Let* $(\mathbb{M}, d)$ *be a metric space and a point* $a$ *of* $\mathbb{M}$*. Then, the singleton set* $\{a\}$ *is a closed set.*

**Proposition 1.12.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $A$ *is open if and only if* $A^c$ *is closed.*

**Proposition 1.13.** *Let* $(\mathbb{M}, d)$ *be a metric space,* $A_1, \ldots, A_n$ *a finite collection of open sets in* $\mathbb{M}$*, and* $B_1, \ldots, B_n$ *a finite collection of closed sets in* $\mathbb{M}$*. Then,* $A_1 \cap \ldots \cap A_n$ *is an open set and* $F_1 \cup \ldots \cup B_n$ *is a closed set.*

**Proposition 1.14.** *Let* $(\mathbb{M}, d)$ *be a metric space,* $\{A_\alpha\}_{\alpha \in I}$ *a collection of open sets in* $\mathbb{M}$ *and* $\{B_\alpha \text{-} \alpha \in I\}$ *(where the index set* $I$ *could be finite, countable, or uncountable). Then, the* $\bigcup_{\alpha \in I} E_\alpha$ *is an open set and* $\bigcap_{\alpha \in I} B_\alpha$ *is a closed set.*

**Proposition 1.15.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $int(A)$ *is the largest open set which is contained in* $A$*; in other words,* $int(A)$ *is open, and given any other open set* $B \subseteq A$*, we have* $A \subseteq int(A)$*. Similarly* $\bar{A}$ *is the smallest closed set which contain* $A$*; in other words,* $\bar{A}$ *is closed, and given any other closed set* $C \supset A, C \supset \bar{A}$*.*

**Proposition 1.16.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $\partial A$ *is a closed set and* $A$ *is closed if and only if* $\partial A \subseteq A$*.*

**Proposition 1.17.** *Let* $(\mathbb{M}, d)$ *be a metric space and* $A$ *a subset of* $\mathbb{M}$*. Then,* $\partial A = \hat{A}\ int(A)$*.*

**Proposition 1.18.** *Let $\{a_k\}$ be a sequence of points in $\mathbb{R}^n$. Then, $\{a_k\}$ has the limit $L$ if and only if each succession $\{a_k^i\}$ of coordinates $i = 1, \ldots, n$ has as a limit the correspondent coordinate $L^j$ of $L$.*

**Proposition 1.19.** *Let $\{a_k\}$ be a sequence of points in $\mathbb{R}^n$. Then, $\{a_k\}$ is bounded if and only if each succession $\{a_k^i\}$ of coordinates $i = 1, \ldots, n$ is bounded.*

**Proposition 1.20.** *Let $a_k$ and $a_k$ two convergent sequence with limits $L_a$ and $L_b$ respectively, and a real number $\lambda$. Then,*

$$\{a_k + b_k\} \to L_a + L_b \qquad \{\lambda a_k\} \to \lambda L_a \tag{12}$$

**Proposition 1.21.** *Let $\{a_k\}$ be a sequence of points in $\mathbb{R}^n$. Then, $\{a_k\}$ is a Cauchy sequence if and only if each succession $\{a_k^i\}$ of coordinates $i = 1, \ldots, n$ has is a Cauchy sequence.*

**Proposition 1.22.** *The set $\mathbb{R}^n$ with the metric $d_{l^2}$ is complete.*

**Definition 1.34.** Let $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $a_i < b_i$ for all $i = 1, \ldots n$. Then, we define the *n-dimensional open interval* $I$ as follows

$$I = \{\vec{x} \in \mathbb{R}^n | a_i < x_i < b_i, i = 1, \ldots, n\} = (a_1, b_1) \times \cdots \times (a_n, b_n). \tag{13}$$

**Definition 1.35.** Let $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $a_i < b_i$ for all $i = 1, \ldots n$. Then, we define the *n-dimensional open interval* $I$ as follows

$$I = \{\vec{x} \in \mathbb{R}^n | a_i \leq x_i \leq b_i, i = 1, \ldots, n\} = [a_1, b_1] \times \cdots \times [a_n, b_n]. \tag{14}$$

**Proposition 1.23.** *Let $I \in \mathbb{R}^n$ be an open interval. Then, $I$ is open.*

**Proposition 1.24.** *Let $I \in \mathbb{R}^n$ be a closed interval. Then, $I$ is closed.*

**Proposition 1.25.** *A set $A \subseteq \mathbb{R}^n$ is compact if and only if is closed and bounded.*

# 2 Functions

**Definition 2.1.** Let $\mathbb{R}^n$ be the set of $n$-tuples of real numbers and $\mathbb{R}$ the set of real numbers, both with the metric $d_{l^2}$. We define a *scalar field* as a function that maps $\mathbb{R}^n$ to $\mathbb{R}$.

$$\begin{aligned} f(\vec{x}) : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{x} &\longmapsto f(\vec{x}) \end{aligned} \tag{15}$$

**Definition 2.2.** Let $\mathbb{R}^n$ be the set of $n$-tuples of real numbers and $\mathbb{R}^m$ be the set of $m$-tuples of real numbers, both with the metric $d_{l^2}$. We define a *vector field* as a function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\begin{aligned} \vec{f}(\vec{x}) : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \vec{x} &\longmapsto \vec{f}(\vec{x}) \end{aligned} \tag{16}$$

**Definition 2.3.** Let be a function $\vec{f}(\vec{x}) : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. We say that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if it satisfies

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta(\varepsilon) \mid 0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \left\| \vec{f}(\vec{x}) - \vec{L} \right\| < \varepsilon \tag{17}$$

in all possible ways.

**Proposition 2.1.** *Let $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\vec{g} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions such that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}_f$ and $\lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) = \vec{L}_g$. Then, the limit of $\vec{f}(\vec{x}) + \vec{g}(\vec{x})$ exists and it satisfies*

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) + \vec{g}(\vec{x}) = \vec{L}_f + \vec{L}_g. \tag{18}$$

**Proposition 2.2.** *Let $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function such that such that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}_f$ and $\lambda$ a real number. Then, the limit of $\lambda \vec{f}(\vec{x})$ exists and it satisfies*

$$\lim_{\vec{x} \to \vec{a}} \lambda \vec{f}(\vec{x}) = \lambda \vec{L}_f. \tag{19}$$

**Proposition 2.3.** *Let $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\vec{g} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions such that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}_f$ and $\lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) = \vec{L}_g$. Then, the limit of $\langle \vec{f}(\vec{x}, \vec{g}(\vec{x}) \rangle_I$ exists and it satisfies*

$$\lim_{\vec{x} \to \vec{a}} \langle \vec{f}(\vec{x}), \vec{g}(\vec{x}) \rangle_I = \langle \vec{L}_f, \vec{L}_g \rangle_I. \tag{20}$$

**Corollary 2.4.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function such that such that $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}_f$. Then, the limit of $\left\| \vec{f}(\vec{x}) \right\|$ exists and it satisfies*

$$\lim_{\vec{x} \to \vec{a}} \left\| \vec{f}(\vec{x}) \right\| = \left\| \vec{L}_f \right\|. \tag{21}$$

**Corollary 2.5.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and a an accumulation point of $\Omega$. Then, it converges in $a$ to the point $L_f$ if and only if each function $f_i(\vec{x})$ of coordinates $i = 1, \ldots, m$ converges to the correspondent coordinates $L_i$ of $L$.*

**Proposition 2.6.** *Let $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\vec{g} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two functions such that $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L_f$ and $\lim_{\vec{x} \to \vec{a}} g(\vec{x}) = L_g$. Then, the limit of $f(\vec{x})/g(\vec{x})$ exists and it satisfies*

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{L_f}{L_g} \tag{22}$$

**Proposition 2.7.** **?** *If $\lim_{(x,y) \to (a,b)} f(x, y) = L$ and exist the following uni-dimensional limits*

$$\lim_{x \to a} f(x, y) \qquad \lim_{y \to b f(x,y)} \tag{23}$$

*then the iterated limits exist and coincide, that is,*

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y) = L. \tag{24}$$

*The reciprocal is not always true.*

**Definition 2.4.** Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $a$ a point of $\Omega$. We say $\vec{f}(\vec{x})$ *is continuous in the point* $a$ if it satisfies the following condition.

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \mid \|\vec{x} - \vec{a}\| < \delta \Rightarrow \left\| \vec{f}(\vec{x}) - \vec{f}(\vec{a}) \right\| < \varepsilon \tag{25}$$

**Proposition 2.8.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and a point of $\Omega$. If a is an accumulation point of $\Omega$, then $\vec{f}$ is continuous in a if and only if*

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a}) \tag{26}$$

**Proposition 2.9.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and a point of $\Omega$. If a is an isolated point of $\Omega$, then $\vec{f}$ is continuous in a.*

**Definition 2.5.** Let $\vec{f} : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $A \subseteq \mathbb{R}^n$ be a set. We say $\vec{f}(\vec{x})$ *is continue in D* if it is continuous in every point $x$ of $A$. In other words, if it satisfies that

$$\forall\varepsilon \in \mathbb{R}^+ \exists\delta(\varepsilon, \vec{x}, \vec{x}') \mid \|\vec{x}' - \vec{x}\| < \delta \Rightarrow \left\|\vec{f}(\vec{x}') - \vec{f}(\vec{x})\right\| < \varepsilon, \forall \vec{x}, \vec{x}' \in A \tag{27}$$

**Proposition 2.10.** *Let $\vec{f} : K \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function, with $K$ a compact set. If f is continuous, then $\vec{f}(K)$ is compact.*

**Proposition 2.11.** *Let $\vec{f} : K \subset \mathbb{R} \longrightarrow \mathbb{R}^m$ be a function, with $K$ a compact set. Then, the maxima and minima of $f$ are in $K$.*

**Proposition 2.12. ?** *Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar continuous function and a an interior point of $D$ such that $f(\vec{a}) \neq 0$. Then, there exists a ball such that all its points have the same sign as $f(\vec{a})$.*

**Definition 2.6.** Let $\vec{f} : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $A \subseteq F$ be a set. We say $\vec{f}(\vec{x})$ *is uniformly continuous in A* if it satisfies the following condition.

$$\forall\varepsilon \in \mathbb{R}^+ \exists\delta(\varepsilon) \mid \|\vec{x} - \vec{x}'\| < \delta \Rightarrow \left\|\vec{f}(\vec{x}) - \vec{f}(\vec{x}')\right\| < \varepsilon \tag{28}$$

**Theorem 2.13.** *Let be $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function in a compact set $A \subset \mathbb{R}^n$. Then, $\vec{f}(\vec{x})$ is uniformly continuous in $A$.*

# 3 Derivative

**Definition 3.1.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, $a$ an interior point of $D$, and $\vec{v}$ a vector of the associated vector space of $\mathbb{R}^n$. Then ,we define the *derivative of $f(\vec{x})$ in a with respect to $\vec{v}$* as

$$f'(a; \vec{v}) := \lim_{h\to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}, \tag{29}$$

provided the limit exists.

**Proposition 3.1.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, a an interior point of $D$, and $\vec{v}$ a vector of $\mathbb{R}^n$. If f is derivable at a, then the derivative is unique.*

**Definition 3.2.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $f'(a; \vec{v})$ the derivative of $f(\vec{x})$ in $a \in D$ with respect to a vector $\vec{v}$. If $\|\vec{v}\| = 1$, then we call it the *directional derivative of $f(\vec{x})$ in a along $\vec{v}$.*

**Theorem 3.2.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, a an interior point of $D$, $\vec{v}$ a vector of $\mathbb{R}^n$, and $g(t) := f(\vec{a} + t\vec{v})$. If one of the derivatives exist, $g'$ or $f'$, the other exists and satisfies that*

$$g'(t) = f'(\vec{a} + t\vec{v}; \vec{v}). \tag{30}$$

*In particular, when $t = 0$ we get $g'(0) = f'(\vec{a}; \vec{v})$.*

**Theorem 3.3** (Intermediate value theorem for scalar functions). *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, a an interior point of $D$, and $\vec{v}$ a vector of $\mathbb{R}^n$. Let us suppose $f(\vec{a} + h\vec{v}; \vec{v})$ is derivable in $[\alpha, \beta]$. Then, there exists a real number $\theta \in (\alpha, \beta)$ such that*

$$\frac{f(\vec{a} + \beta\vec{y}) - f(\vec{a} + \alpha\vec{v})}{\beta - \alpha} = f'(\vec{a} + \theta\vec{v}, \vec{v}) \tag{31}$$

*In particular, when $\alpha = 0$ we have $f(\vec{a} + \beta\vec{y}) - f(\vec{a}) = \beta f'(\vec{a} + \theta\vec{v}, \vec{v})$.*

**Definition 3.3.** revisar Let $a$ be an interior point an open set $\Omega \subseteq \mathbb{R}^n$, $f(\vec{x}_i) : S \longrightarrow \mathbb{R}$ a scalar function and $B = (e_1, \dots, e_n)$ an orthonormal base the associated vector space $\mathbb{R}^n$ such that $\vec{x} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$. Then, we define the *partial derivative of $f(\vec{x})$ in a point a with respect to $\vec{e}_i$* as the directional derivative (if exists) of $f(\vec{x})$ in $a$ along $\vec{e}_i$.

$$\frac{\partial f(\vec{a})}{\partial x_i} := f'(\vec{a}, \vec{e}_i) \tag{32}$$

We also denote it by $D_i f(\vec{x})$, $\partial_{x_i} f(\vec{x})$, and $f'_{x_i}(\vec{x})$.

**Definition 3.4.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $a$ an interior point of $D$. We say $f$ *is differentiable at a* if and only if there exists a linear function $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\lim_{\vec{v}\to\vec{0}} \frac{|f(\vec{a} + \vec{v}) - f(\vec{a}) - L(\vec{v})|}{\|\vec{v}\|} = 0. \tag{33}$$

**Theorem 3.4.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, a an interior point of $D$, and $\vec{v}$ a vector of $\mathbb{R}^n$. If $f(\vec{x})$ is differentiable at a, then there exist all the directional derivatives in a and it satisfies*

$$f'(\vec{a}; \vec{v}) = L(\vec{v}) = \sum_{i=1}^{n} v_i \frac{\partial f(\vec{a})}{\partial x_i} \tag{34}$$

**Theorem 3.5.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and a an interior point of $D$. If f is differentiable at a, then f is continuous at a.*

**Theorem 3.6.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and a an interior point of $D$. Let $\partial f(\vec{a})/\partial x_1, \dots, \partial f(\vec{a})/\partial x_n$ be its partial derivatives in a. If they are continuous, then f is differentiable in a.*

**Proposition 3.7.** *Let be the following functions.*

$$\begin{aligned} \vec{r} : \Omega_1 \subseteq \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \vec{r}(t) \end{aligned} \qquad \begin{aligned} f : \Omega_2 \supseteq f(\Omega_1) &\longrightarrow \mathbb{R} \\ \vec{r}(t) &\longmapsto f(\vec{r}(t)) \end{aligned}, \tag{35}$$

*with $\Omega_1, \Omega_2$ open sets. If $\vec{r}(t)$ exist*

**Definition 3.5.** Let $f(\vec{x}) : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. We define the *differential of f* as the limit of $\Delta f(\vec{x})$ when $\|\Delta\vec{x}\|$ tends to 0.

**Proposition 3.8.** *Let $f : \Omega_1 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \Omega_2 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be two functions, with $\Omega_1, \Omega_2$ open set, and a an interior point of $\Omega_1$ and $\Omega_2$. If $f, g$ are differentiable in a, then $f + g$ is differentiable in a and $d(f + g) = df + dg$.*

**Proposition 3.9.** *Let $f : \Omega_1 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \Omega_2 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be two functions, with $\Omega_1, \Omega_2$ open set, and a an interior point of $\Omega_1$ and $\Omega_2$. If $f, g$ are differentiable in a, then $fg$ is differentiable in a and $d(fg) = gdf + fdg$.*

**Definition 3.6.** Let $\vec{f}(\vec{x}) : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a point $a$ an interior point of $D$.

**Proposition 3.10.** *Let $\vec{f}(\vec{x}) : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function, a a point in D,*

**Definition 3.7.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $a$ an interior point of $D$. We say $f$ *is differentiable at a* if and only if there exists a linear function $J : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{\vec{v}\to\vec{0}} \frac{\|f(\vec{a}+\vec{v}) - f(\vec{a}) - J(\vec{v})\|}{\|\vec{v}\|} = 0. \tag{36}$$

**Proposition 3.11.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and a an interior point of D. Then, $\vec{f}$ is differentiable at a if and only if all components of $\vec{f}$ are differentiable at a.*

**Theorem 3.12.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function in an interior point a of D. Then, $\vec{f}'(\vec{a};\vec{v}) = \hat{T}\vec{v}$ and, if $\vec{f} = (f_1, \ldots, f_m)$ and $\vec{v} = (y_1, \ldots, y_n)$ (with orthonormal basis for $\mathbb{R}^m$ and $\mathbb{R}^n$), then*

$$\hat{T}\vec{v} = \vec{f}'(\vec{a};\vec{v}) = \sum_{i=1}^{m} \langle \vec{\nabla} f_i(\vec{a}), \vec{v}\rangle_I \vec{e}_i \tag{37}$$

**Theorem 3.13.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function and a an interior point of D. If $\vec{f}$ is differentiable in a, then $\vec{f}$ is continuous in a.*

**Proposition 3.14.** *Let $\vec{f} : \Omega_1 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\vec{g} : \Omega_2 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions, with $\Omega_1, \Omega_2$ open set, and a an interior point of $\Omega_1$ and $\Omega_2$. If $\vec{f}, \vec{g}$ are differentiable in a, then $\vec{f} + \vec{g}$ is differentiable in a and $d(\vec{f} + \vec{g}) = d\vec{f} + d\vec{g}$.*

**Proposition 3.15.** *Let $\vec{f} : \Omega_1 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\vec{g} : \Omega_2 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions, with $\Omega_1, \Omega_2$ open set, and a an interior point of $\Omega_1$ and $\Omega_2$. If $\vec{f}, \vec{g}$ are differentiable in a, then $\vec{f}\vec{g}$ is differentiable in a and $d(\vec{f} + \vec{g}) = \vec{g}d\vec{f} + \vec{f}d\vec{g}$.*

**Proposition 3.16.** *Let $\vec{f}(\vec{x}) : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function in a point a in D.*

**Proposition 3.17.** *Let $\vec{g} : A \subseteq \mathbb{R}^l \longrightarrow \mathbb{R}^n$ and $\vec{f} : B \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions. Let a be an interior point of A, and let $b = \vec{g}(\vec{a})$ be an interior point of B. If $\vec{g}$ is differentiable at a and $\vec{f}$ differentiable at b, then $(\vec{f} \circ \vec{g})$ is differentiable at a and*

$$\left[(\vec{f} \circ \vec{g})(\vec{a})\right] = \left[\vec{f}'(\vec{g}(\vec{a}))\right] \left[\vec{g}'(\vec{a})\right]. \tag{38}$$

**Proposition 3.18.** *Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be two functions. Let a be an interior point of A and $f(a) = b$ an interior point of B. If $\vec{g}$ is differentiable at b, $\vec{g} \circ \vec{f}$ differentiable at a, and if the jacobian of g is an injective linear function, then $[g'(b)]^{-1}$ and $[\vec{f}'(\vec{a})]$ exist, and*

$$[\vec{f}'(a)] = [g'(\vec{f}(\vec{a}))]^{-1} \circ (\vec{g} \circ \vec{f})'(\vec{a}). \tag{39}$$

**Theorem 3.19.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and a, b two interior points of D. If $(1-t)a + tb$ are interior point of D for all $(0, 1)$, and if f is differentiable at all these points, then there exists a point $c = (1-\tau)a + \tau b$ for some $\tau \in (0, 1)$ such that*

$$f(\vec{b}) - f(\vec{a}) = \langle \vec{\nabla} f(\vec{c}), \vec{b} - \vec{a}\rangle_I. \tag{40}$$

**Corollary 3.20.** *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function with $\Omega$ an open set such that, whenever $a, b \in \Omega$, then $(1-t)a + tb \in \Omega$ for all $t \in (0, 1)$ ($\Omega$ is convex). If $[\vec{f}'(\vec{x})]$ exists and is the zero function for all $x \in \Omega$, then $\vec{f}(x) = \vec{c}$, where $\vec{c}$ is a constant vector.*

**Definition 3.8.** Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar function and $\Lambda \subseteq \Omega$, with $\Omega$ an open set. We say $f$ *is a function of class $C^k(\Lambda)$* with $k \in \mathbb{N}$ if its partial derivatives until $k$-th degree exist and are continuous in $\Lambda$.

**Definition 3.9.** Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar function and $\Lambda \subseteq \Omega$, with $\Omega$ an open set. We say $f$ *is a function of class $C^\infty(\Lambda)$* if it is of class $C^k(\Lambda)$ for all $k \in \mathbb{N}$.

**Proposition 3.21.** *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function with $\Omega \subseteq \mathbb{R}^n$ an open set.*

**Theorem 3.22.** *Let $\vec{f} : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and a a point in D. If $\partial^2\vec{f}(\vec{a})/\partial x_i \partial x_j$ and $\partial\vec{f}(\vec{a}/\partial x_j \partial x_i)$ exist and one of them is continuous in a, then the other is also continuous in a and*

$$\frac{\partial f}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial f}{\partial x_j \partial x_i}(\vec{a}) \tag{41}$$

**Theorem 3.23.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and a an interior point of D. If for some $\delta$, $\partial_{x_i} f(\vec{x}), \partial_{x_j} f(\vec{x}), \partial_{x_i}\partial_{x_j} f(\vec{x})$ exist in $B(a, \delta)$ and if $\partial_{x_i}\partial_{x_j} f(\vec{x})$ is continuous in a, then $\partial_{x_j}\partial_{x_i} f(\vec{x})$ exists and*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{a}). \tag{42}$$

**Definition 3.10.** Let $f : \Omega \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ be a function of class $C^2(\Omega)$, with $\Omega$ an open set and $a$ a point of $\Omega$. Then, we define the *Hessian matrix of $f$ at a point $a$* as the following matrix.

$$\begin{pmatrix} \frac{\partial^2 f(\vec{a})}{\partial x_i^2} & \cdots & \frac{\partial^2 f(\vec{a})}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\vec{a})}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 f(\vec{a})}{\partial x_m^2} \end{pmatrix} \tag{43}$$

**Theorem 3.24.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and a an interior point of D. If for some*

$\delta$, $\partial_{x_i}\vec{f}(\vec{x}), \partial_{x_j}\vec{f}(\vec{x}), \partial_{x_i}\partial_{x_j}\vec{f}(\vec{x})$ *exist in* $B(a,\delta)$ *and if* $\partial_{x_i}\partial_{x_j}\vec{f}(\vec{x})$ *is continuous in* $a$, *then* $\partial_{x_j}\partial_{x_i}\vec{f}(\vec{x})$ *exists and*

$$\frac{\partial^2 \vec{f}}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial^2 \vec{f}}{\partial x_j \partial x_i}(\vec{a}). \tag{44}$$

# 4 Maxima and minima

**Proposition 4.1.** *Let a function of class* $C^2(D)$ *and* $a$ *an interior point of* $D$. *Then,*

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \frac{h^T \nabla f(\vec{a})}{1!} + \frac{h^T H(\vec{a}+\theta\vec{h})h}{2!},\ \theta \in (0,1) \tag{45}$$

*is equivalent to*

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \frac{h^T \nabla f(\vec{a})}{1!} + \frac{h^T H(\vec{a})h}{2!} + \|\vec{h}\|^2 E_2(\vec{a};\vec{h}). \tag{46}$$

**Definition 4.1.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $a$ a point of $D$. We say $f$ *has a local maxima in* $a$ if there exists a ball $B(a,r) \subset D$ such that

$$f(\vec{x}) \geq f(\vec{a}), \forall x \in B(a,r) \tag{47}$$

**Definition 4.2.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $a$ a point of $D$. We say $f$ *has a local minima in* $a$ if there exists a ball $B(a,r) \subset D$ such that

$$f(\vec{x}) \leq f(\vec{a}), \forall x \in B(a,r) \tag{48}$$

**Definition 4.3.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $a$ a point of $D$. We say $f$ *has a global maxima in* $a$ if

$$f(\vec{x}) \leq f(\vec{a}), \forall x \in D \tag{49}$$

**Definition 4.4.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $a$ a point of $D$. We say $f$ *has a global minima in* $a$ if

$$f(\vec{x}) \geq f(\vec{a}), \forall x \in D \tag{50}$$

**Proposition 4.2.** *Let* $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a function and* $a$ *an interior point of* $D$. *If* $f$ *is differentiable in* $a$ *and has a local extreme in* $a$, *then* $\vec{\nabla} f = \vec{0}$ *(which is equivalent to say that every component, that is, every partial derivative, is 0).*

**Definition 4.5.** Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field, with $\Omega$ an open set, and $a$ a point in $\Omega$. We say *$a$ is a stationary or extreme point* if $\vec{\nabla}(\vec{a}) = \vec{0}$.

**Definition 4.6.** Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field, with $\Omega$ an open set, and $a$ a point in $\Omega$. We say *$a$ is a saddle point* if

$$\forall B(a,r) \exists \vec{x} \mid f(\vec{x}) > f(\vec{a}) \wedge \exists \vec{y} \mid f(\vec{y}) < f(\vec{a}) \tag{51}$$

**Proposition 4.3.** *Let* $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a scalar field of class* $C^2$, *with* $\Omega$ *an open set. Let* $Q(h)$ *quadratic form associated to the Hessian matrix of* $f$ *in an arbitrary point of* $\Omega$. *Then, there are two real numbers* $m, M$ *such that*

$$m\|h\|^2 \leq Q(h) \leq M\|h\|^2 \tag{52}$$

**Proposition 4.4** (Criterion of sufficiency of stationary points)**.** *Let* $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a scalar field of class* $C^2$, *with* $\Omega$ *an open set, and* $a$ *an stationary point of* $f$ *in* $\Omega$. *If the Hessian matrix of* $f$ *is positively (negatively) defined in* $a$, *then* $a$ *is a minima (maxima) of* $f$.

**Theorem 4.5.** *Let* $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a scalar field of class* $C^2$, *with* $\Omega$ *an open set, and* $a$ *an stationary point of* $f$ *in* $\Omega$. *Then,*

- *If* $\det\{H(\vec{a})\} > 0$ *and* $\partial^2 f(\vec{a})/\partial x^2 > 0$, *then* $f$ *has a local minima in* $a$.
- *If* $\det\{H(\vec{a})\} > 0$ *and* $\partial^2 f(\vec{a})/\partial x^2 < 0$, *then* $f$ *has a local maxima in* $a$.
- *If* $\det\{H(\vec{a})\} < 0$, *then* $f$ *has a saddle point in* $a$.
- *If* $\det\{H(\vec{a})\} = 0$, *we can't determine the point.*

# 5 Implicit function

**Theorem 5.1** (Inverse function Theorem)**.** *Let* $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a function with* $\Omega$ *an open set. If* $\vec{f} \in C^1(\Omega)$, *and if, for some point* $a \in \Omega$, $\operatorname{rank}[\vec{f}'(\vec{a})] = n$, *the there exist open sets* $U, V \subseteq \mathbb{R}^n$ *such that* $a \in U, \vec{f}(\vec{a}) \in V$, *and*

1. $\vec{f} : U \longrightarrow V$ *is injective,*
2. $\vec{f}^{-1} : V \longrightarrow U$ *exists,*
3. $\vec{f}^{-1} \in C^1(V)$,
4. $\det\Big\{[\vec{f}(\vec{x})]\Big\} \neq 0$ *for all* $x \in U$, *and*
5. $\det\Big\{[\vec{f}^{-1}(\vec{x})]\Big\} \neq 0$ *for all* $y \in V$.

**Theorem 5.2** (Theorem of the implicit function)**.** *Let* $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a scalar field of class* $C^1$, *with* $\Omega$ *an open set and* $a$ *a point in* $\Omega$ *such that* $f(\vec{a}) = 0$ *and* $\partial f/\partial x_n(\vec{a}) \neq 0$. *Then, there exists a ball* $B(\vec{a};r) \subset \Omega$, *an open set* $\Gamma \subseteq \mathbb{R}^{n-1}$ *that contains* $a$, *and a function* $g : \Gamma \longrightarrow \mathbb{R}$ *of class* $C^1(\Gamma)$ *such that*

$$x \in B(\vec{a};r) \qquad \text{and} \qquad f(\vec{x}) = 0 \tag{53}$$

*if and only if*

$$(x_1, \ldots, x_{n-1}) \in \Gamma \qquad \text{and} \qquad x_n = g(x_1, \ldots, x_{n-1}). \tag{54}$$

**Proposition 5.3.** *Let* $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a scalar field of class* $C^2$, *with* $\Omega$ *an open set, and* $(x_0, y_0)$ *a point in* $\Omega$ *such that* $f((x_0,y_0)) = 0$ *and* $\partial f/\partial xy((x_0,y_0)) \neq 0$. *Then, there exist a rectangle* $R = [x_0-\alpha, x_0+\alpha] \times [y_0-\beta, y_0+\beta]$ *such that for every* $x \in R$ *the equation* $F(x,y) = 0$ *there is a unique solution* $y = g(x)$ *where* $y \in [y_0-\beta, y_0+\beta]$.

**Proposition 5.4.** *If* $y = g(x)$, *then* $g$ *is continuous, its derivative* $g'$ *is continuous and*

$$g'(x) = -\frac{\partial_x f(x, g(x))}{\partial_y f(x, g(x))} \tag{55}$$

**Theorem 5.5** (Implicit function Theorem). *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a function with $\Omega$ an open set and $\vec{f}$ of class $C^1(\Omega)$. If there is a point $(a,b) \in \Omega$ such that $\vec{f}(\vec{a},\vec{b}) = 0$ and*

$$\det\left\{J_{\vec{f},\vec{x}}(\vec{a},\vec{b})\right\} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a},\vec{b}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{a},\vec{b}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{a},\vec{b}) & \dots & \frac{\partial f_n}{\partial x_n}(\vec{a},\vec{b}) \end{vmatrix} \neq 0, \tag{56}$$

*then there exists an open set $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $(a,b)$, an open set $V \subseteq \mathbb{R}^m$ containing $b$, and a function $\vec{g} : V \longrightarrow \mathbb{R}^n$ such that $\vec{h}(\vec{b}) = \vec{a}$ and $\vec{f}(\vec{h}(\vec{y}),\vec{y}) = 0$ for all $y \in V$. Furthermore, $\vec{g} \in C^1(V)$ and $\vec{g}$ is uniquely determined by $(g(y),y) \in U$ for all $y \in V$.*

**Theorem 5.6.** *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of the form $u = f(x_1,\dots,x_n)$ where the variables are conditioned by $m < n$ equations $\phi_1 = \dots = \phi_m = 0$, with $\phi_i : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ a function as $f$ defined in an open set $\Omega$. If $f, \phi_1, \dots, \phi_m$ are functions of class $C^2(\Omega)$ and not every $\det\{[\phi_i]\}$ with respect $m$ of the $n$ variables is zero at the extreme, then by introducing $\lambda_1,\dots,\lambda_m$ Lagrange multipliers and making the $n+m$ derivatives of $F(\lambda_1,\dots,\lambda_m,x_1,\dots,x_n) = f + \lambda_1\phi_1 + \cdots \lambda_m\phi_m$ equal to zero, we obtain the equations*

$$\frac{\partial F}{\partial \lambda_1} = \dots = \frac{\partial F}{\partial \lambda_m} = \frac{\partial F}{\partial x_1,} = \dots = \frac{\partial F}{\partial x_n} = 0. \tag{57}$$

*These equations form a system of $n+m$ equations for the $n+m$ variables $\lambda_1,\dots,\lambda_m,x_1,\dots,x_n$ that must satisfy the extreme of $f$.*

# 6 Integrals

**Definition 6.1.** Let $\Pi_j = \{x_0^{(j)},\dots,x_{k_j}^{(j)}\}$ with $x_0^{(j)} = a_j$, $x_{k_j}^{(j)} = b_j$, and $j = 1,\dots,n$ be sets that form partitions of $[a_j,b_j]$. Then, we define the *partition of the $n$-dimensional interval $I$* as

$$\Pi = \Pi_1 \times \dots \times \Pi_n. \tag{58}$$

**Definition 6.2.** Let $I$ be a closed $n$-dimensional interval, $f : I \longrightarrow \mathbb{R}$ be a bounded function, and $\Pi$ a partition of $I$ which divides $I$ into $\mu$ $n$-dimensional closed intervals $I_1,\dots,I_\mu$. Then, we define, we define the following notations

$$m_k(f) \coloneqq \inf_{x\in I_k} f(x), \qquad M_k(f) \coloneqq \sup_{x\in I_k} f(x). \tag{59}$$

**Definition 6.3.** Let $I$ be a closed $n$-dimensional interval, $f : I \longrightarrow \mathbb{R}$ be a bounded function, and $\Pi$ a partition of $I$ which divides $I$ into $\mu$ $n$-dimensional closed intervals $I_1,\dots,I_\mu$. Then, we define the *lower sum of $f$ on $I$* as

$$\underline{S}(f;\Pi) \coloneqq \sum_{k=1}^{\mu} m_k(f)|I_k|. \tag{60}$$

**Definition 6.4.** Let $I$ be a closed $n$-dimensional interval, $f : I \longrightarrow \mathbb{R}$ be a bounded function, and $\Pi$ a partition of $I$ which divides $I$ into $\mu$ $n$-dimensional closed intervals $I_1,\dots,I_\mu$. Then, we define the *upper sum of $f$ on $I$* as

$$\bar{S}(f;\Pi) \coloneqq \sum_{k=1}^{\mu} M_k(f)|I_k|. \tag{61}$$

**Definition 6.5.** Let $\Pi, \Pi'$ be two partitions of an $n$-dimensional interval $I$. We say $\Pi'$ *is a refinement of* $\Pi$, $\Pi' \supseteq \Pi$, if and only if every point in $\Pi$ is also a point in $\Pi'$.

**Proposition 6.1.** *Let $I$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a bounded function. Then,*

1. *$\underline{S}(f;\Pi) \leq \bar{S}(f;\Pi)$, for every partition $\Pi$ of $I$,*
2. *$\underline{S}(f;\Pi) \leq \underline{S}(f;\Pi'), \bar{S}(f;\Pi) \geq \bar{S}(f;\Pi')$, if $\Pi' \supseteq \Pi$,*
3. *$\underline{S}(f;\Pi) \leq \bar{S}(f;\Pi')$ for any partitions $\Pi, \Pi'$ of $I$.*

**Definition 6.6.** Let $I$ be an $n$-dimensional closed interval, $f : I \longrightarrow \mathbb{R}$ a bounded function, and $\Pi$ a partition of $I$. Then, we define the *lower integral* and *upper integral*, respectively, as

$$\underline{\int_I} f \coloneqq \sup_{(\Pi)} \underline{S}(f;\Pi), \qquad \overline{\int_I} f \coloneqq \inf_{(\Pi)} \bar{S}(f;\Pi). \tag{62}$$

**Proposition 6.2.** *Let $I$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a bounded function. Then, the lower and upper integral exist and*

$$\underline{\int_I} f \leq \overline{\int_I} f. \tag{63}$$

**Definition 6.7.** Let $f : I \longrightarrow \mathbb{R}$ be a function. We say $f$ *is Riemann-integrable on $I$* if and only if

$$\underline{\int_I} f = \overline{\int_I} f. \tag{64}$$

In this case, call it the *Riemann integral of $f$ over $I$* and denote it by

$$\int_I f. \tag{65}$$

**Theorem 6.3. ?** *Let $I$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a bounded function. Then, $f$ is Riemann-integrable on $I$ if and only if it satisfies the following condition*

$$\forall\varepsilon > 0 \exists\ \Pi(\varepsilon) \mid \bar{S}(f;P) - \underline{S}(f;P) < \varepsilon, \forall\Pi \supset \Pi(\varepsilon). \tag{66}$$

**Proposition 6.4.** *Let $I$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a continuous function. Then, $f$ is Riemann-integrable.*

**Definition 6.8.** Let $S \subseteq \mathbb{R}^n$ be a set. We say *$S$ has an $n$-dimensional Lebesgue measure zero*, $\lambda(S) = 0$, if and only if one can find, for every $\varepsilon > 0$, a sequence of $n$-dimensional open intervals $\{I_k\}$ such that

1. $S \subseteq \bigcup_{k\in\mathbb{N}} I_k$,
2. $\sum_{k=1}^{\infty} |I_k| < \varepsilon$.

**Theorem 6.5.** *Let $I$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a function that is continuous on $I$ except on a set of $n$-dimensional Lebesgue measure zero. Then, $f$ is Riemann-integrable.*

**Proposition 6.6.** *Let $I$ be an $n$-dimensional closed interval, $f, g : I \longrightarrow \mathbb{R}$ two Riemann-integrable functions and $\lambda, \mu$ two real numbers. Then, $\lambda f + \mu g$ is Riemann-integrable and*

$$\int_I \lambda f + \mu g = \lambda \int_I f + \mu \int_I g. \tag{67}$$

**Definition 6.9.** Let $D \subseteq \mathbb{R}^n$ be a bounded set and $f : D \longrightarrow \mathbb{R}$ a bounded function. Then, we define

$$f_D(x) = \begin{cases} f(x), & \text{if } x \in D, \\ 0, & \text{if } x \notin D. \end{cases} \tag{68}$$

**Proposition 6.7. ?** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a bounded function with $D$ a bounded set. Let $f_D : I \longrightarrow \mathbb{R}$ be the extension of $F$ in an $n$-dimensional closed interval $I$ that contains $D$. If the Riemann integral of $f_D$ on $I$ exists, then*

$$\int_I f_D = \int_{I'} f_D \tag{69}$$

*for all $n$-dimensional intervals $I'$ that contain $D$.*

**Definition 6.10.** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a bounded function with $D$ a bounded set. Then, we say $f$ *is Riemann-integrable over $D$* if and only if $f_D$ is Riemann-integrable over (at least) one $n$-dimensional interval $I \supseteq D$. In that case,

$$\int_D f := \int_I f_D. \tag{70}$$

**Theorem 6.8.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a bounded function with $D$ a bounded set. If $\lambda(\partial D) = 0$ and $f$ is continuous almost everywhere on $D$, then $\int_D f$ exists.*

**Definition 6.11.** Let $D \subseteq \mathbb{R}^n$ be a bounded set and $I$ an $n$-dimensional closed interval such that $I \supseteq D$. We define the *characteristic function $\chi_D : \mathbb{R}^n \to \mathbb{R}$ of $D$* as the following function

$$\chi_D(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{if } x \notin D. \end{cases} \tag{71}$$

**Definition 6.12.** Let $D \subseteq \mathbb{R}^n$ be a bounded set, $I$ an $n$-dimensional closed interval such that $I$, and $\Pi$ a partition on $I$. We define the *inner $n$-dimensional Jordan content of $D$* as

$$\underline{J}(D) := \sup_{(\Pi)} \underline{S}(\chi_D, \Pi) \tag{72}$$

**Definition 6.13.** Let $D \subseteq \mathbb{R}^n$ be a bounded set, $I$ an $n$-dimensional closed interval such that $I$, and $\Pi$ a partition on $I$. We define the *outer $n$-dimensional Jordan content of $D$* as

$$\underline{J}(D) := \inf_{(\Pi)} \bar{S}(\chi_D, \Pi) \tag{73}$$

**Definition 6.14. ?** Let $D \subseteq \mathbb{R}^n$ be a bounded set, $I$ an $n$-dimensional closed interval such that $I$, and $\Pi$ a partition on $I$. We say $D$ *has $n$-dimensional Jordan content $J(D)$* if and only if

$$\underline{J}(D) = \bar{J}(D). \tag{74}$$

Then, $J(D) = \underline{J}(D) = \bar{J}(D)$. A bounded set which has Jordan content is called *Jordan-measurable*.

**Theorem 6.9.** *Let $D \subseteq \mathbb{R}^n$ be a bounded set and $I$ an $n$-dimensional closed interval such that $I$. Then, $D$ is Jordan-measurable if and only if the characteristic function $\chi_D$ of $D$ is integrable on $I$. In that case,*

$$J(D) = \int_I \chi_D. \tag{75}$$

**Theorem 6.10.** *Let $D \subseteq \mathbb{R}^n$ be a bounded set and $I$ an $n$-dimensional closed interval such that $I$. Then, $D$ is Jordan-measurable if and only if $\partial D$ has Jordan content zero.*

**Proposition 6.11.** *Let $I, J \subseteq \mathbb{R}^n$ be two closed intervals and let $f : I \times J \to \mathbb{R}$ be a bounded function. If we define $\phi_x : J \to \mathbb{R}$ by $\phi_x(y) = f(x, y)$, then*

$$\underline{\Phi}(x) = \underline{\int_J} \phi_x, \qquad \bar{\Phi}(x) = \overline{\int_J} \phi_x \tag{76}$$

*exist for all $x \in I$. If $\Pi_I, \Pi_I'$ are two partitions of $I$ and $\Pi_B, \Pi_B'$ two partitions of $B$, then*

$$\underline{S}(f;\Pi) \leq \underline{S}(\underline{\Phi};\Pi_I) \leq \underline{S}(\bar{\Phi};\Pi_I) \leq \bar{S}(\bar{\Phi};\Pi_I') \leq \bar{S}(f;\Pi'), \tag{77}$$

*where $\Pi = \Pi_A \times \Pi_b$ and $\Pi' = \Pi_A' \times \Pi_B'$. If we define $\psi_y ; I \to \mathbb{R}$ by $\psi_y = f(x, y)$, then*

$$\underline{\Psi}(y) = \underline{\int_I} \psi_y, \qquad \bar{\Psi}(y) = \overline{\int_I} \psi_y \tag{78}$$

*exist for all $y \in J$ and*

$$\underline{S}(f;\Pi) \leq \underline{S}(\underline{\Psi};\Pi_J) \leq \underline{S}(\bar{\Psi};\Pi_J) \leq \bar{S}(\bar{\Psi};\Pi_J') \leq \bar{S}(f;\Pi'). \tag{79}$$

**Theorem 6.12** (Fubini's Theorem)**.** *Let $I, J \in \mathbb{R}^n$ be two closed intervals. If $f : I \times J \to \mathbb{R}$ is integrable on $I \times J$, then*

$$\int_{I\times J} f = \int_I \underline{\Phi} = \int_I \bar{\Phi} = \int_J \underline{\Psi} = \int_J \bar{\Psi}. \tag{80}$$

**Theorem 6.13.** *Let $I, J \subseteq \mathbb{R}^n$ be two closed intervals. If $f : I \times J \to \mathbb{R}$ is integrable on $I \times J$, then*

1. *if $\phi_x$ is integrable on $J$, then*

$$\int_{I\times J} f = \int_I \int_J \phi_x, \tag{81}$$

2. *if $\psi_y$ is integrable on $I$, then*

$$\int\limits_{I\times J} f = \int\limits_J \int\limits_I \psi_y, \tag{82}$$

3. *and if both $\phi_x, \psi_y$ are integrable on $I, J$ (that is, $f$ is continuous on $I \times J$), then*

$$\int\limits_{I\times J} f = \int\limits_I \int\limits_J \phi_x = \int\limits_J \int\limits_I \psi_y, \tag{83}$$

*or more explicitly,*

$$\int\limits_{I\times J} f(\vec{x},\vec{y})d\hat{x}d\hat{y} = \int\limits_I \int\limits_J f(\vec{x},\vec{y})d\hat{y}d\hat{x} = \int\limits_J \int\limits_I f(\vec{x},\vec{y})d\hat{x}d\hat{y}. \tag{84}$$

**Corollary 6.14.** *Let $I = [a_1, b_1] \times \cdots [a_n, b_n] \subseteq \mathbb{R}^n$ be an $n$-dimensional closed interval and $f : I \longrightarrow \mathbb{R}$ a continuous function. Then,*

$$\int\limits_I f = \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} f(\vec{x})dx_1 \dots dx_n. \tag{85}$$

**Theorem 6.15.** *Let $\alpha, \beta : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions such that $\alpha(x) \leq \beta(x)$ for all $x \in [a, b]$ and let*

$$D = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\right\}. \tag{86}$$

*If $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then the Riemann integral of $f$ on $D$ exists and*

$$\int\limits_D f = \int\limits_a^b \int\limits_{\alpha(x)}^{\beta(x)} f(x, y)dydx. \tag{87}$$

**Theorem 6.16.** *Let $\alpha, \beta, \gamma, \delta$ be four continuous functions, and let*

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, a \leq x \leq b, \alpha(x) \leq y \leq \beta(x), \gamma(x, y) \leq z \leq \delta(x, y)\right\}. \tag{88}$$

*If $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, the the Riemann integral of $f$ on $D$ exists and*

$$\int\limits_D f = \int\limits_a^b \int\limits_{\alpha(x)}^{\beta(x)} \int\limits_{\gamma(x,y)}^{\delta(x,y)} f(x, y, z)dzdydx. \tag{89}$$

# 7 Line integrals

**Theorem 7.1.** *Let $f(x, y)$ be a continuous function in a rectangle $[\alpha, \beta] \times [a, b]$. Then, the integral $F(x) = \int_a^b f(x, y)dy$ is a continuous function in $[\alpha, \beta]$.*

**Theorem 7.2.** *Let $f(x, y)$ be a continuous function in region delimited by $\psi_1(x)$ and $\psi_2(x)$, with $x \in [\alpha, \beta]$ and $\psi_1, \psi_2$ continuous functions. Then, the integral $F(x) = \int_a^b f(x, y)dy$ is a continuous function in $[\alpha, \beta]$.*

**Theorem 7.3.** *Let $R = [\alpha, \beta] \times [a, b]$ be a rectangle and $f : R \longrightarrow \mathbb{R}$ a Riemann-integrable function on $[a, b]$ for all $y \in [c, d]$. If $\partial_y f$ is continuous on $R$, then the function $F : [c, d] \longrightarrow \mathbb{R}$ is of class $C^1[c, d]$ and*

$$\frac{dF}{dy} = \frac{d}{dy} \int\limits_a^b f(x, y)dx = \int\limits_a^b \frac{\partial f}{\partial y}(x, y)dx. \tag{90}$$

**Theorem 7.4.** *Let $f(x, y)$ be a continuous function in $[\alpha, \beta] \times [a, b]$. If the function has $\partial_x f$ continuous, then*

$$\frac{dF}{dx} = \frac{d}{dx} \int\limits_a^b f(x, y)dy = \int\limits_a^b \frac{\partial f}{\partial x}(x, y)dy \tag{91}$$

*and $F'$ is continuous.*

**Theorem 7.5.** *Let $F(x) = \int_{\psi_1(x)}^{\psi_2(x)} f(x, y)dy$ and $\psi_1, \psi_2$ are of class $C^1(I)$, with $I = [\alpha, \beta]$. If $f$ and $\partial_x f$ are continuous in $[\alpha, \beta] \times [a, b]$, then*

$$\frac{dF}{dx} = \frac{d}{dy} \int\limits_{\psi_1(x)}^{\psi_2(x)} f(x, y)dy = \tag{92}$$

$$\int\limits_{\psi_1(x)}^{\psi_2(x)} \frac{\partial f}{\partial x}(x, y)dy + f(x, \psi_2(x))\psi_2'(x) - f(x, \psi_1(x))\psi_1'(x). \tag{93}$$

**Theorem 7.6.** *Let $f(x, y)$ be a continuous function defined in the rectangle $[\alpha, \beta] \times [a, b]$. Then,*

$$\int\limits_a^b \int\limits_\alpha^\beta f(x, y)dxdy = \int\limits_\alpha^\beta \int\limits_a^b f(x, y)dydx \tag{94}$$

**Definition 7.1.** Let $\Gamma \subseteq \mathbb{R}^n$ be a set. We say $\Gamma$ *is a curve in* $\mathbb{R}^n$ if and only if there exists an interval $I \subseteq \mathbb{R}$ and a continuous function $\gamma : I \longrightarrow R^n$ such that $\mathrm{Im}\,(\gamma) = \Gamma$. In that case, we call $\vec{r} : I \longrightarrow \Gamma$ a *parametrization of* $\Gamma$. We call

$$\begin{aligned} x_1 &= \phi_1(t) \\ &\vdots \\ x_n &= \phi_n(t) \end{aligned} \tag{95}$$

the *parameter representation of* $\Gamma$ and $t$ the *parameter*.

**Definition 7.2.** Let $\Gamma \subseteq \mathbb{R}^n$ be a set. We say $\Gamma$ *is a simple curve in* $\mathbb{R}^n$ if and only if there exists an interval $I \subseteq \mathbb{R}$ and a continuous injective function $\gamma : I \longrightarrow \mathbb{R}^n$ such that $\mathrm{Im}\,(\gamma) = \Gamma$. In that case, we call $\gamma : I \longrightarrow \Gamma$ an *injective parametrization of* $\Gamma$.

**Definition 7.3.** Let $\Gamma \subseteq \mathbb{R}$ be a curve with a parametrization $\gamma : I \longrightarrow \Gamma$ and $\varphi : J \longrightarrow I$ a diffeomorphism. Then, $\gamma \circ \varphi : J \longrightarrow \Gamma$ is a *reparametrization of* $\Gamma$.

**Definition 7.4.** Let $\Gamma \subseteq \mathbb{R}^n$ be a curve with a parametrization $\gamma : I \longrightarrow \Gamma$. We say $\Gamma$ *is regular* if it is differentiable and its parametrization $\gamma$ never vanishes.

**Definition 7.5.** Let $\Gamma \subseteq \mathbb{R}^n$ be a curve with a parametrization $\gamma : I \longrightarrow \Gamma$. We say $\Gamma$ *is piecewise-regular* if

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [t_1, t_2] \\ \vdots & \\ \gamma_m(t), & t \in [t_n, t_{n+1}] \end{cases}, \tag{96}$$

with $\gamma_1, \ldots, \gamma_m$ being regular curves.

**Definition 7.6.** Let $\mathbb{C}$ be a curve. Then, we define an arc of a curve as a function $\alpha : [a,b] \longrightarrow \mathscr{C}$.

**Definition 7.7.** Let $\alpha$ be an arc of a curve $\mathscr{C}$. Then, we say $\vec{\alpha}$ is a simple curve if $\vec{\alpha}$ is bijective.

**Definition 7.8.** Let $\alpha$ be an arc of a curve $\mathscr{C}$. Then, we say $\vec{\alpha}$ is a regular curve if $\vec{\alpha}$ is of class $C^1$ in pieces, that is, the discontinuity is only present in a finite number of points.

**Definition 7.9.** Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a vector field. Let be a simple and regular arc of a curve $\mathscr{C}$. Then, we define the *line integral over the path* $\mathscr{C}$ as

$$\int\limits_{\mathscr{C}} \langle \vec{f}, d\vec{r} \rangle_I := \lim_{\Pi} \sum_{i=1} \langle \vec{f}(\vec{x}_i'), \vec{x}_{i+1} - \vec{x}_i \rangle_I, \qquad \vec{x}_i' \text{ between } \vec{x}_{i+1} - \vec{x}_i \tag{97}$$

**Definition 7.10.** Let $\Gamma \subseteq \mathbb{R}^n$ be a curve of class $C^1(I)$ and let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function. Then, we define the *line integral of* $\vec{f}$ *along* $\Gamma$ as

$$\int\limits_{\vec{f}} \langle :=, \mathrm{d}\mathbf{r} \rangle_I \int\limits_a^b \langle \vec{f}(\gamma(t)), \gamma'(t) \rangle_I dt. \tag{98}$$

**Proposition 7.7.** *Let ... Then,*

$$\int\limits_{\mathscr{C}} \langle \lambda \vec{f}(\vec{\alpha}) + \mu \vec{g}(\vec{\alpha}), d\vec{\alpha} \rangle_I = \lambda \int\limits_{\mathscr{C}} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I + \mu \int\limits_{\mathscr{C}} \langle \vec{g}(\vec{\alpha}), d\vec{\alpha} \rangle_I \tag{99}$$

**Proposition 7.8.** *Let... Then,*

$$\left| \int\limits_{\mathscr{C}} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I \right| \leq \int\limits_a^b \left| \left\langle \vec{f}(\vec{\alpha}(t)), \frac{d\vec{\alpha}}{dt}(t) \right\rangle_I \right| dt \tag{100}$$

**Proposition 7.9** (Additivity)**.** *Let* $\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2$ *... Then,*

$$\int\limits_{\mathscr{C}} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I = \int\limits_{\mathscr{C}_1} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I + \int\limits_{\mathscr{C}_2} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I. \tag{101}$$

**Proposition 7.10.** *The line integral depend on the orientation*

$$\int\limits_{\mathscr{C}_{A\to B}} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I = - \int\limits_{\mathscr{C}_{B\to A}} \langle \vec{f}(\vec{\alpha}), d\vec{\alpha} \rangle_I \tag{102}$$

**Proposition 7.11.** *If the parametrization preserve the orientation of the curve, the line integral does not depend on the parametrization*

**Definition 7.11.** For scalar fields

$$\int\limits_{\mathscr{C}} \varrho(\vec{x}) dl, \tag{103}$$

where $dl$ is the length differential.

**Definition 7.12.** Let $\Gamma \subseteq \mathbb{R}^n$ be a curve with a parametrization $\gamma \in C^1(I)$ and $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ a continuous function. Then, we define the *line integral of* $f$ *along* $\Gamma$ as

$$\int\limits_{f} := \mathrm{d}r \int\limits_a^b f(\gamma(t)) \left\| \gamma'(t) \right\| dt. \tag{104}$$

**Definition 7.13.** Let $\Gamma \subseteq \mathbb{R}$ be a curve with a parametrization $\vec{r} : I \longrightarrow \Gamma$ and $\Pi$ a partition of $I$. Then, we define the *length of the polygonal* as

$$L(\vec{r}; \Pi) := \sum_{i=1}^n \| \vec{r}(t_i) - \vec{r}(t_{i-1}) \|. \tag{105}$$

**Definition 7.14.** Let $\Gamma \subseteq \mathbb{R}$ be a curve with a parametrization $\vec{r} : I \longrightarrow \Gamma$ and $\Pi$ a partition of $I$. Then, we define the length of $\Gamma$ as

$$L(\Gamma) = \sup_{\Pi} L(\vec{r}; \Pi). \tag{106}$$

**Definition 7.15.** Let $\Gamma \subseteq \mathbb{R}$ be a curve. We say *the curve* $\Gamma$ *is rectifiable* if and only if its length is finite.

**Proposition 7.12.** *Let* $\Gamma \subseteq \mathbb{R}$ *be a curve with a parametrization* $\vec{r} : I \longrightarrow \Gamma$*. If* $\vec{r} \in C^1(I)$*, then* $\Gamma$ *is rectifiable and*

$$L(\Gamma) = \int\limits_a^b \| \vec{r}'(t) \| dt. \tag{107}$$

**Definition 7.16.** Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say $S$ *is connected* if and only if it cannot be represented as the union of two or more disjoint non-empty open subsets. If not, we say $S$ *is disconnected*

**Definition 7.17.** Let $(\mathbb{M}, d)$ be a metric space and $S \subseteq \mathbb{M}$ a set. We say $S$ *is path-connected* if every pair of points can be connected by a continuous path that belongs to the set.

**Definition 7.18.** A *path* is a continuous function $f : [0,1] \longrightarrow \mathbb{R}^n$. We call $f(0)$ the *initial point* and $f(a)$ the *terminal point*.

**Definition 7.19.** Let $S$ be a connected set. We say *it is simply connected* if it is path-connected and every path between two points can be continuously transformed into any other such path while preserving the two endpoints in question. Equivalently, $S$ is simply connected if it is path connected and any loop in $S$ can be contracted to a point. Otherwise, we say it is multiply connected.

**Definition 7.20.** Let $S$ be a simply connected set. We say $S$ *is convex* if for all pair of points $a, b \in S$, the segment defined by

$$[a,b] = \{x \mid x = (1-t)a + tb, 0 \leq t \leq 1\} \tag{108}$$

is contained in $S$, that is, if every pair of points can be connected by a straight line that belongs to the set.

**Theorem 7.13** (Gradient Theorem)**.** *Let $\Gamma \subseteq \mathbb{R}^n$ be curve with a parametrization $\gamma : [a,b] \longrightarrow \Gamma$ and $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ a differentiable function with $D$ a connected set. Then,*

$$\int_{\vec{\nabla} f} \langle =, \mathrm{d}\mathbf{r}\rangle_I f(\gamma(b)) - f(\gamma(a)). \tag{109}$$

**Corollary 7.14.** *Let $\Gamma \subseteq \mathbb{R}^n$ be closed curve with a parametrization $\gamma : [a,b] \longrightarrow \Gamma$ and $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ a differentiable function with $D$ a connected set. Then,*

$$\oint_{\Gamma} \langle \vec{\nabla} f, d\vec{r}\rangle_I = 0. \tag{110}$$

**Theorem 7.15.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function, with $\Omega$ a connected and open set. If the line integral of $\vec{f}$ between two pints is independent on the curve and, given a point $a \in \Omega$ we define the function*

$$\varphi(\vec{x}) = \int_a^x \langle \vec{f}, d\vec{r}\rangle_I = \int_{\gamma} \langle \vec{f}, \mathrm{d}\mathbf{r}\rangle_I \tag{111}$$

*with $\Gamma$ an arbitrary piece-wise regular. Then,*

$$\vec{f} = \vec{\nabla}\varphi, \qquad \forall x \in \Omega. \tag{112}$$

**Theorem 7.16.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class $C^0(\Omega)$ (continuous), with $\Omega$ a connected an open set. Then, the following conditions are equivalent.*

1. *$\vec{f} = \vec{\nabla}\varphi$ for some $\varphi(\vec{x})$.*
2. *The line integral of $\vec{f}$ does not depend on the path*
3. *The line integral of $\vec{f}$ over every piece-wise regular closed path contained in $\Omega$ is zero.*

**Theorem 7.17.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a function of class $C^1(D)$, with $D$ not necessarily connected. If $\vec{f} = \vec{\nabla}\varphi$, then*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}. \tag{113}$$

*In particular, in $\mathbb{R}^3$, if $\vec{f} = \vec{\nabla}\varphi$ then $\vec{\nabla} \times \vec{f} = \vec{0}$.*

**Theorem 7.18.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function of class $C^1(\Omega)$ with $\Omega$ a convex and open set. Then, $\vec{f} = \vec{\nabla}\varphi$ if and only if*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \qquad \forall \vec{x}_i \in \Omega. \tag{114}$$

**Definition 7.21.** Let $[a,b] \times [c,d]$ be a closed rectangle. If we make a partition $\Pi_1$ in $[a,b]$ and $\Pi_2$ in $[c,d]$, then we define the partition of the rectangle as a collection of sets $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

**Definition 7.22.** Let $f$ be a fuction. We say $f$ *is scalonated* if there is a partition of a rectangle $R$ such that $f$ is constant in every point of the open rectangles that define the partition.

**Proposition 7.19.** *If $f$ and $g$ are scalonated functions, then $c_1 f + c_2 g$ is a scalonated function.*

**Definition 7.23.** Let $f$ be a scalonaed function and $R = [a,b] \times [c,d]$ a rectangle of domain of $f$. Then, given a partition $\Pi$ over $R$, we define the subrectangle $r$ as $[x_{i-1}, x_i] \times y_{j-1}, y_j$. In each subrectangle, the function behaves as a constant $c_{ij}$. Then, we define

$$\iint_R f = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}(x_i - x_{i-1})(y_j - y_{j-1}). \tag{115}$$

**Proposition 7.20.** *Let $f$ be a function defined in a rectangle $R$. Then,*

$$\iint_R f = \int_a^b \int_c^d f dydx = \int_c^d \int_a^b f dxdy \tag{116}$$

**Proposition 7.21.**

$$\int_a^b \int_c^d \lambda f + \mu g dydx = \lambda \int_a^b \int_c^d f dydx + \mu \int_a^b \int_c^d g dydx \tag{117}$$

**Proposition 7.22.** *If we have a rectangle $R$ that is $R_1 \cup R_2$. Then,*

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f \tag{118}$$

**Theorem 7.23** (Theorem of comparation)**.** *If $R \subseteq T$, then*

$$\iint_R f \leq \iint_T f \tag{119}$$

**Definition 7.24.** Let $f$ be a bounded fnction, that is, $|f| \leq M$. Let $s$ and $t$ two scalonated functions such that $s \leq f \leq t$ (that exist because $f$ is bounded). If there is a unique number $I$ such that

$$\iint_R s \leq \iint_R f \leq \iint_R t$$

for every pair of functions that satisfy the condition we presented before, then $I$ is called double integral of $f$ over the rectangle $R$ and it is denoted by

$$\iint_R f(x,y)dxdy. \tag{120}$$

**Definition 7.25.** Let $S$

$$S := \left\{ \iint_R s \,\middle|\, s \leq f, \forall (x,y) \in R \right\} \tag{121}$$

**Definition 7.26.** Let $S$

$$T := \left\{ \iint\limits_R t \middle| t \geq f, \forall (x,y) \in R \right\} \tag{122}$$

**Theorem 7.24.** *We say $f$ is integrable if the supreme of $S$ and the infime of $T$ are equal.*

**Proposition 7.25.** *Additivity, ... LOOK IN THE PDF MORE*

**Theorem 7.26.** *Let $f : R \longrightarrow \mathbb{R}$ be an integrable function in the rectangle $R = [a,b] \times [c,d]$. Let us suppose there exist*

$$A(y) = \int\limits_a^b f(x,y)dx, \forall y \in [c,d], \qquad \int\limits_c^d A(y)dy.$$

*Then,*

$$\iint\limits_R f = \int\limits_c^d \int\limits_a^b f(x,y)dxdy \tag{123}$$

**Corollary 7.27.** *Integrating first by $y$ and then by $x$ is the same.*

**Theorem 7.28.** *Let $f : R \longrightarrow \mathbb{R}$ be a continuous function defined in a rectangle $R = [a,b] \times [c,d]$. Then, $f$ is integrable in $R$ and the integral can be calculated by iterated integrals.*

**Definition 7.27.** Let $A \subseteq \mathbb{R}^2$ be a bounded set. We say $A$ *has content/measurement/area null* if $\forall \varepsilon > 0$ there is a finite covering of $A$ with rectangles of area $< \varepsilon$.

**Theorem 7.29.** *Let $f$ be a bounded function defined in $R = [a,b] \times [c,d]$. If the set of discontinuities of $f$ has measure zero, then $f$ is integrable in $R$.*

**Theorem 7.30.** *Let $\varphi(x)$ be a continuous function defined in $[a,b]$. Then, the graph of $\varphi(x)$ is a set of measure zero.*

**Theorem 7.31.** *Let $S$ be a region of first kind. Let $f$ be a continuous function define in $S$. Then, the double integral of $f$ in $S$ exists and*

$$\iint\limits_S f = \int\limits_a^b \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dydx. \tag{124}$$

**Theorem 7.32.** *If $S$ is of second kind, then the integral exists and*

$$\iint\limits_S f = \int\limits_c^d \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy. \tag{125}$$

**Definition 7.28.** A *Jordan curve* $\Gamma \subseteq \mathbb{R}^n$ is a closed simple piece-wise regular curve.

**Theorem 7.33.** *Let $P, Q : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ two functions of class $C^1(D)$- Let $\Gamma \subseteq D$ be a Jordan curve and $R = \Gamma \cup \operatorname{int}\Gamma$ (which is simply connected). If $R$ is a region of first and second kind, then*

$$\iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \oint\limits_{\partial R} Pdx + Qdy \tag{126}$$

**Corollary 7.34.** *The area can be calculated as*

$$\oint\limits_{\partial R} xdy = \oint\limits_{\partial R} -ydx \tag{127}$$

*and infinite other always that $\partial_x Q - \partial_y P = 1$.*

**Theorem 7.35.** *Let $\vec{f} : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by $\vec{f} = P\vec{e}_x + Q\vec{e}_y$ and of class $C^1(\Omega)$, with $\Omega$ a simply connected and open set. Then, $\vec{f} = \vec{\nabla}\varphi$ if and only if*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{128}$$

**Theorem 7.36** (Green's Theorem for multiply connected regions)**.** *Let $\Gamma_1, \ldots, \Gamma_k$ be Jordan curves such that*

1. $\Gamma_i \cap \Gamma_j = \emptyset$,
2. $\forall i, \Gamma_i \subseteq \operatorname{int}\Gamma_1$,
3. $\forall i \neq j \geq 2 \Gamma_i \subseteq \operatorname{ext}\Gamma_j \Leftrightarrow \operatorname{int}\Gamma_i \cap \operatorname{int}\Gamma_j = \emptyset$.

*Let $R = \Gamma_1 \cup \operatorname{int}\Gamma_1 - \bigcup_{i=2}^k \operatorname{int}\Gamma_i$ (which is multiply connected if $k \geq 2$). Let $\vec{f} : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by $\vec{f} = P\vec{e}_x + Q\vec{e}_y$ and of class $C^1(\Omega)$, with $\Omega \supseteq R$ a connected and open set. Then,*

$$\iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \oint\limits_{\Gamma_1} Pdx + Qdy - \sum_{i=2}^n \oint\limits_{\Gamma_i} Pdx + Qdy \tag{129}$$

**Theorem 7.37** (Invariance over path deformation)**.** *Let $P, Q : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ two functions of class $C^1(\Omega)$ with $\Omega$ a connected and open set and such that $\partial_y P = \partial_x Q$ for all point of $\Omega$. Let $\Gamma_1, \Gamma_2 \in \Omega$ be two Jordan curves such that*

1. $\Gamma_2 \subseteq \operatorname{int}\Gamma_2$,
2. $\operatorname{int}\Gamma_1 \cap \operatorname{ext}\Gamma_2 \subseteq \Omega$.

*Then,*

$$\oint\limits_{\Gamma_1} Pdx + Qdy = \oint\limits_{\Gamma_2} Pdx + Qdy$$

**Theorem 7.38.** *Let $f$ be a function in $\Omega$. Let $u = \phi(x,y)$ and $v = \psi(x,y)$ be two bijective functions of class $C^1(\Omega)$ and such that $\partial(\phi,\psi)/\partial(x,y) \neq 0$. Then.*

$$\iint\limits_\Omega f(x,y)dxdy = \iint\limits_\Upsilon f(\phi^{-1}(u,v), \psi^{-1}(u,v)) \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| dudv. \tag{130}$$

**Theorem 7.39** (Jacobi's Theorem)**.** **?** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $D$ a Jordan-measurable set such that $D \cup \partial D \subseteq \Omega$. If $g : \Omega \longrightarrow \mathbb{R}^n$, $g \in C^1(\Omega)$, $g : \operatorname{int}(D) \longrightarrow \mathbb{R}^n$, $J[g(x)] \neq 0$ for all $x \in \operatorname{int}(D)$, then $g(D)$ is Jordan-measurable and its Jordan content is given by*

$$J(g(D)) = \int\limits_D \|J[\vec{g}(\vec{x})]\| \, d\hat{x}. \tag{131}$$

*If, in addition, $f : g(D) \longrightarrow \mathbb{R}$ is bounded and continuous, then*

$$\int\limits_{\vec{g}(D)} f(\vec{y}) d\hat{y} = \int\limits_D (f \circ \vec{g})(\vec{y}) \, \|J[g(\vec{y})]\| \, d\hat{y}. \tag{132}$$

*Or more explicitly, with $g(\vec{y}) = (\phi_1, \dots, \phi_n)$ and $\phi_i = \phi_i(\vec{y})$,*

$$\int\limits_{\vec{g}(D)}\!\!\cdots\!\!\int f(x_1, \dots, x_n) dx_1 \dots dx_n = \int\limits_D\!\!\cdots\!\!\int f(\phi_1, \dots, \phi_n) \left\| \begin{matrix} \partial_{y_1}\phi_1 & \dots & \partial_{y_n}\phi_1 \\ \vdots & \ddots & \vdots \\ \partial_{y_1}\phi_n & \dots & \partial_{y_n}\phi_n \end{matrix} \right\| dy_1 \dots dy_n \tag{133}$$

**Definition 7.29.** let $V \subseteq \mathbb{R}^3$. We say $V$ *is projectable in xy* if there exists the following set

$$V_{xy} = \{(x, y) \in S | \phi_1(x, y) \leq z \leq \phi_2(x, y)\},$$

with $\phi_1, \phi_2$ continuous functions.

**Definition 7.30.**

$$R_{xz} = \{(x, z) \in S | \psi_1(x, z) \leq y \leq \psi_2(x, z)\} \tag{134}$$

**Definition 7.31.**

$$R_{yz} = \{(y, z) \in S | \psi_1(y, z) \leq x \leq \psi_2(y, z)\} \tag{135}$$

# 8 Differential geometry

**Definition 8.1.** **?** Let $\Sigma \subseteq \mathbb{R}^3$ be a set. We say $\Sigma$ *is a surface* if and only if there exists a connected set $\Omega \subseteq \mathbb{R}^3$ with nonempty interior and a continuous function $\vec{r} : \Omega \longrightarrow \mathbb{R}^3$ such that $\operatorname{Im}(f) = \Sigma$. In this case, we call $\sigma : \Omega \longrightarrow \Sigma$ *the parametrization of* $\Sigma$. We call

$$\begin{aligned} x_1 &= \phi_1(u, v), \\ x_2 &= \phi_2(u, v), \\ x_3 &= \phi_3(u, v) \end{aligned} \tag{136}$$

*the parameter representation of* $\Sigma$ and $(u, v)$ the *parameter.*

**Definition 8.2.** **?** Let $\Sigma \subseteq \mathbb{R}^3$ be a set. We say $\Sigma$ *is a simple surface* if and only if there exists a connected set $\Omega \subseteq \mathbb{R}^3$ with nonempty interior and a injective continuous function $\sigma : \Omega \longrightarrow \mathbb{R}^3$ such that $\operatorname{Im}(f) = \Sigma$. In this case, we call $\sigma : \Omega \longrightarrow \Sigma$ *the injective parametrization of* $\Sigma$.

**Definition 8.3.** Let $\Sigma$ be a surface. Then, we define the *area of* $\Sigma$ as

$$\iint\limits_\Omega \left\| \frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u} \right\| dudv \tag{137}$$

**Definition 8.4.** Let $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function and a surface $\Sigma \subseteq D$ with a parametrization $\sigma : \Omega \longrightarrow \Sigma$ of class $C^1(\Omega)$. If $f$ is bounded in $\Sigma$, then we define the *surface integral of $f$* as

$$\iint\limits_\Sigma f ds := \iint\limits_\Omega f \left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| dudv. \tag{138}$$

**Theorem 8.1.** *Under these conditions, the integral does not depend on the parametrization.*

**Definition 8.5.** Let $\vec{f} : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function and a surface $\Sigma \subseteq D$ with a parametrization $\sigma : \Omega \longrightarrow \Sigma$ of class $C^1(\Omega)$. If $\vec{f}$ is bounded in $\Sigma$, then we define the *surface integral of $f$* as

$$\int\limits_\Sigma \langle \vec{f}, \mathbf{ds} \rangle_I := \iint\limits_\Omega \left\langle \vec{f}, \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\rangle_I dudv \tag{139}$$

**Theorem 8.2** (Stokes' Theorem)**.** *Let $\vec{f} : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function of class $C^1$ in a simple surface $\Sigma \subseteq D$ (with a parametrization $\sigma : \Omega \longrightarrow \Sigma$ of class $C^2(\Omega)$ in all points except in sets of measure zero), $\Omega$ of first and second kind, and the Jordan curve $\Gamma = \partial\Sigma$ piece-wise regular. Then,*

$$\int\limits_\Sigma \langle \vec{\nabla} \times \vec{f}, \mathbf{ds} \rangle_I = \oint\limits_{\partial\Sigma} \langle \vec{f}, d\vec{r} \rangle_I, \tag{140}$$

*where the direction of the surface vector is defined by the right-hand rule with respect the direction of rotation of the curve.*

**Theorem 8.3** (Stokes' Theorem for non-convex sets)**.** *Let $S = \vec{r}(T)$ a simple surface with $T$ a plane region of first and second kind, with $\vec{r}$ of class $C^2$ except of sets of measure zero and the $S$ being bounded by a Jordan curve $\Gamma$ regular in pieces. Let $\vec{F} = (P, Q, R)$ defined in $S$ of class $C^1$. Then,*

$$\int\limits_\Sigma \langle \vec{\nabla} \times \vec{F}, d\vec{s} \rangle_I = \oint\limits_{\partial\Sigma} \langle \vec{F} \cdot d\vec{\alpha} \rangle_I - \oint\limits_C \langle \vec{F} \cdot d\vec{\alpha} \rangle_I, \tag{141}$$

*where the direction of the surface vector is defined by the right-hand rule with respect the direction of rotation of the curve and $C$ is the boundary of the hole.*

**Theorem 8.4.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function of class $C^1(D)$ and let $B$ be an orthonormal basis. Then,*

- *If $D$ is convex, then $\partial_i f_j = \partial_j f_i \Leftrightarrow \vec{f} = \vec{\nabla}\phi$ [ ],*
- *If $D$ is simply connected and $m = 2$, then $\partial_i f_j = \partial_j f_i \Leftrightarrow \vec{f} = \vec{\nabla}\phi$ [ ],*
- *If $D$ is simply connected and $m = 3$, then $\partial_i f_j = \partial_j f_i \Leftrightarrow \vec{f} = \vec{\nabla}\phi$.*

**Theorem 8.5** (Gauss' Theorem)**.** *Let $V \subseteq \mathbb{R}^3$ be a symmetric projectable solid that is limited by an orientable surface and $\vec{f} : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ a function of class $C^1(D)$. If $d\vec{s}_{ext}$ is the exterior differential of surface, then*

$$\iiint_V \langle \vec{\nabla}, \vec{f} \rangle_I dv = \oiint_{\partial V} \langle \vec{f}, d\vec{s}_{ext} \rangle_I. \tag{142}$$

**Theorem 8.6.** *Let $\vec{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function of class $C^1$ in a closed ball $B(a,t)$. Then,*

$$\langle \vec{\nabla}, \vec{f}(a) \rangle_I = \lim_{t \to 0} \frac{1}{|V(t)|} \oiint_{\partial V(t)} \langle \vec{f}, d\vec{s}_{ext} \rangle_I. \tag{143}$$

**Theorem 8.7.** *Let $\vec{f} : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function of class $C^1$ in a closed disc $\Sigma(t) \subseteq D$. Then*

$$\langle \vec{n}, \vec{\nabla} \times \vec{f}(a) \rangle_I = \lim_{t \to 0} \frac{1}{|\Sigma(t)|} \oint_{\partial \Sigma(t)} \langle \vec{f}, d\vec{r} \rangle_I. \tag{144}$$

*where*

- *The surface is arbitrary*
- *The curve is the boundary of $S$*
- *$\vec{n}$ is a unitary vector perpendicular to $S$*
- *the integral is done with the right-hand rule and the vector $\vec{n}$*

**Theorem 8.8.** *With the same conditions as the Gauss' theorem, then*

$$\iiint_V \vec{\nabla} \times \vec{f} dv = -\oiint_{\partial V} \vec{f} \times d\vec{s}_{ext}. \tag{145}$$

**Proposition 8.9.** *We have*

- $\langle \vec{\nabla}, \vec{f} \times \vec{n} \rangle_I = \langle \vec{n}, \vec{\nabla} \times \vec{f} \rangle_I - \langle \vec{f}, \vec{\nabla} \times \vec{n} \rangle_I$
- $\langle \vec{n}, \vec{f} \times d\vec{s} \rangle_I = \langle d\vec{s}, \vec{n} \times \vec{f} \rangle_I = \langle \vec{f}, d\vec{s} \times \vec{n} \rangle_I$

**Theorem 8.10.** *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ two functions and a an interior point of $D$. Then,*

$$\vec{\nabla} \times \vec{\nabla} f = \vec{0}, \qquad \langle \vec{\nabla}, \vec{\nabla} \times \vec{g} \rangle_I = 0. \tag{146}$$