1 Vector analysis

Definition 1.1. Let $\mathfrak{R} = (O, \mathfrak{B})$ be a reference the euclidean space $\mathbb{R}^3$ and P a point of the this affine space. We define the position vector $\vec{r}_p$ of the point P as the point that satisfies

$$P = O + \vec{r}_p \tag{1}$$

2 Electrostatics

Law 1 (Coulomb's Law)**.** *Let q_1, q_2 be two charges at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then,*

$$\mathbf{F}_{1\to 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\|\mathbf{r}_2 - \mathbf{r}_1\|^2}\mathbf{e}_{1\to 2}. \tag{2}$$

Proposition 2.1. *Let $\mathbf{F}$ be the electric force. Then, $\mathbf{F}$ satisfies the strong form of Newton's Third Law.*

Axiom 1 (Superposition principle)**.** *The total force is the sum of forces caused by every individual charge.*

Proposition 2.2. *Let $\mathbf{r}_p \in \mathbb{R}^3$ be a vector, $B(r_p, \rho)$ an open ball of center r_p and radius δ, and $\lambda \in \mathbb{R}$ a number. Then, the integral*

$$I = \int\limits_{B(r_p,\delta)} \frac{1}{\|\mathbf{r} - \mathbf{r}_q\|^\lambda}\,\mathrm{d}v_q \tag{3}$$

is convergent for $\lambda < 3$ and divergent for $\lambda \geq 3$ **?***.*

Corollary 2.3. *Let $\mathbf{r}_p \in \mathbb{R}^3$ be a position vector, $K \subseteq \mathbb{R}^3$ a compact, and $\rho(\mathbf{r}_p) : K \longrightarrow \mathbb{R}$ a bounded function. Then, the integral*

$$I = \int\limits_{K} \rho(\mathbf{r}_p)\frac{\mathbf{r}_p - \mathbf{r}_1}{\|\mathbf{r}_p - \mathbf{r}_1\|^3}\,\mathrm{d}v_q \tag{4}$$

is convergent **?***.*

Corollary 2.4. *Let $\mathbf{r}_p \in \mathbb{R}^3$ be a position vector and $\rho(\mathbf{r}_p) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ a function of class $C^\infty(\mathbb{R}^3)$ and a compact support. Then, the electric field can be calculated by the integral* **?**

$$\mathbf{E} = \int\limits_{\mathbb{R}^3} \rho(\mathbf{r}_p)\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|^2}\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|}\,\mathrm{d}v_q\,. \tag{5}$$

Proposition 2.5. *Let $\mathbf{r}_p = (x^1, x^2, x^3) \in \mathbb{R}^3$ be a position vector, $\rho(\mathbf{r}_p) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ a function of class $C^\infty(\mathbb{R}^3)$ and a compact support, and the integral*

$$\Phi(\mathbf{r}) = \int\limits_{\mathbb{R}^3} \rho(\mathbf{r}_p)\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|}\,\mathrm{d}v_q\,. \tag{6}$$

Then, the integral is convergent, the function is derivable and its partial derivatives are obtained deriving under the integral sign **?**

$$\frac{\partial\Phi}{\partial x^i} = \int\limits_{\mathbb{R}^3} \frac{\partial}{\partial x^i}\left(\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|}\right)\rho(\mathbf{r}_q)\,\mathrm{d}v_q\,. \tag{7}$$

Law 2 (Gauss's Law)**.** *Let $\mathbf{E}$ be the electric field produced by Coulomb's Law. If it satisfies the superposition principle, then*

$$\boxed{\oint\limits_{\partial V} \langle \mathbf{E}, \mathrm{d}\mathbf{s}\rangle_I = \frac{Q_{\text{int}}}{\epsilon_0}.} \tag{8}$$

Definition 2.1. Potential

$$\Phi(\mathbf{r}) := \int\limits_{\mathbf{r}_0}^{\mathbf{r}} \langle \mathbf{E}(\mathbf{r}'), \mathrm{d}\mathbf{r}'\rangle_I, \qquad [\Phi] = \mathrm{V} := \mathrm{C\,m} = \mathrm{J\,C^{-1}} \tag{9}$$

Theorem 2.6.

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{10}$$

Proposition 2.7. *Let Φ be the potential generated by $\mathbf{E}$. Then, Φ obeys the superposition principle.*

Theorem 2.8. *$\partial\Phi/\partial x^i$ is derivable with respect to x^i and*

$$\frac{\partial^2\Phi}{\partial (x^i)^2} = \int\limits_{\mathbb{R}^3} \frac{\partial}{\partial x^i}\left(\frac{\rho(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|^2}\frac{x_q^i - x_p^i}{\|\mathbf{r}_p - \mathbf{r}_q\|}\right)\mathrm{d}v_q - \rho(\mathbf{r}_p)\int\limits_{S^2} (\omega^i)^2\,\mathrm{d}\omega\,, \tag{11}$$

which is convergent and where $S^2 = \{\mathbf{r}_q \in \mathbb{R}^3 \mid \|\mathbf{r}_q\| = 1\}$ **?***.*

Theorem 2.9 (Earnshaw's Theorem for Electrostatics)**.** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone* **?***.*

Proposition 2.10. *Let $\Phi(x, y)$ be the solution of the Laplace equation in two dimensions and Γ a curve described by the condition $(x - x_0)^2 + (y - y_0)^2 = R^2$. Then,*

$$\Phi(x_0, y_0) = \frac{1}{2\pi R}\oint\limits_{\Gamma} \Phi\,\mathrm{d}r\,. \tag{12}$$

Proposition 2.11. *Let $\Phi(x, y, z)$ be the solution of the Laplace equation in three dimensions and Σ a surface described by the condition $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$. Then,*

$$\Phi(\mathbf{r}_0) = \frac{1}{4\pi R^2}\oint\limits_{\Sigma} \Phi\,\mathrm{d}s\,. \tag{13}$$

Theorem 2.12 (Thomson's Theorem)**.** *If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors* **?***.*

Theorem 2.13. *The electrostatic energy of a point charge q near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is half the Coulombic energy between the charge q and each image charge* **?***.*

Theorem 2.14. *The electrostatic energy of a set of m point charges $q_1, \ldots, q_m$ near n perfect conductors of arbitrary shapes, each conductor being either neutral or grounded, is the Coulombic interaction energy between the real point charges plus half the sum, from $i = 1$ to $i = m$, of the Coulombic energies between charge q_i and each image charge* **?**.

3 Electrostatics in matter

Definition 3.1. Dipolar electric moment

$$\mathbf{p} \coloneqq \int_{\Omega} \mathbf{r}\rho(\mathbf{r})\,\mathrm{d}v\,, \qquad [p] = \mathrm{C\,m}. \tag{14}$$

Theorem 3.1 (Green's Reciprocity Theorem)**.** *Let $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ two charge distributions producing two potentials $\Phi_1(\mathbf{r}), \Phi_2(\mathbf{r})$ respectively. Then,*

$$\boxed{\int_{\mathbb{R}^3} \rho_1\Phi_2\,\mathrm{d}v = \int_{\mathbb{R}^3} \rho_2\Phi_1\,\mathrm{d}v\,.} \tag{15}$$

Definition 3.2. Electric polarization vector,

$$\mathbf{P} \coloneqq \frac{1}{\Delta V}\sum_{i=1}^{n} \mathbf{p}_i, \qquad [P] = \mathrm{C\,m^{-2}} \tag{16}$$

Definition 3.3. Bounded charge or polarization charge

$$\rho_p \coloneqq -\operatorname{div}\mathbf{P}, \qquad [\rho_p] = \mathrm{C\,m^{-3}}, \tag{17}$$

$$\sigma_p \coloneqq \langle \mathbf{p}, \mathbf{n}\rangle_I, \qquad [\sigma_p] = \mathrm{C\,m^{-2}}. \tag{18}$$

Proposition 3.2. *In the interior and exterior of a dielectric,*

$$\mathbf{E}(\mathbf{r}_p) = \frac{1}{4\pi\epsilon_0}\oint_{\partial\Omega} \left\langle \sigma_p(\mathbf{r}_q)\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}, \mathrm{d}\mathbf{s}\right\rangle_I + \frac{1}{4\pi\epsilon_0}\int_{\Omega} \rho_p(\mathbf{r}_q)\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}\,\mathrm{d}v \tag{19}$$

4 Magnetostatics

Definition 4.1. Density current

$$\mathbf{J} \coloneqq \rho\mathbf{v}, \qquad [J] = \mathrm{A\,m^{-2}}; \mathbf{K} \coloneqq \sigma\mathbf{v}, \qquad [K] = \mathrm{A\,m^{-1}}. \tag{20}$$

Definition 4.2. Electric current

$$I \coloneqq \int_{\sigma} \langle \mathbf{J}, \mathrm{d}\mathbf{s}\rangle_I, \int_{\gamma} \langle \mathbf{K}, \mathrm{d}\mathbf{r}\rangle_I, \qquad [I] = \mathrm{C\,m^{-1}} = \mathrm{A}. \tag{21}$$

Axiom 2 (Ohm's law)**.**

$$\mathbf{J} = g\mathbf{E}, \qquad [g] = \Omega^{-1}\,\mathrm{m^{-1}} = \mathrm{S\,m^{-1}}, \tag{22}$$

where g is called the conductivity.

Proposition 4.1 (Ohm's macroscopic law)**.**

$$\Delta V = IR, \qquad R = \frac{\rho l}{s}, \qquad [\rho] = \Omega\,\mathrm{m}, \tag{23}$$

where ρ is called the resistivity.

Proposition 4.2 (Joule's law)**.**

$$P = IV. \tag{24}$$

Theorem 4.3 (Continuity equation)**.**

$$\operatorname{div}\mathbf{J} + \frac{\partial\rho}{\partial t} = 0. \tag{25}$$

Theorem 4.4 (Kirchhoff's circuit laws)**.** *1.*

2. For intensity,

$$\sum_{i=1}^{n} I_i = 0. \tag{26}$$

3. For voltage,

$$\sum_{i=1}^{n} V_i = 0. \tag{27}$$

Axiom 3.

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}_q)\times\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}\,\mathrm{d}v_q\,, \qquad [B] = \mathrm{Wb\,m^{-2}} = \mathrm{T} = \mathrm{N} \tag{28}$$

Definition 4.3. Magnetic permeability

$$\mu_0 \coloneqq \frac{1}{\epsilon_0 c^2}, \qquad [\mu_0] = \mathrm{N\,A^{-2}}. \tag{29}$$

Axiom 4.

$$\mathbf{F}(\mathbf{r_p}) = \int_V \mathbf{J}(\mathbf{r}_p)\times\mathbf{B}(\mathbf{r}_p)\,\mathrm{d}v_p \tag{30}$$

Axiom 5 (Lorentz Force)**.**

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}; \qquad \mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi}q\mathbf{v}\times\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}. \tag{31}$$

Theorem 4.5.

$$\mathbf{F}_{1\to 2} = \frac{\mu_0}{4\pi}I_1 I_2\int_{\gamma_1}\int_{\gamma_2} \mathrm{d}\mathbf{r}_2\times\left(\mathrm{d}\mathbf{r}_1\times\frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3}\right) = -\frac{\mu_0}{4\pi}I_1 I_2\int_{\gamma_1}\int_{\gamma_2} \tag{32}$$

Corollary 4.6. $\mathbf{F}_{1\to 2} = -\mathbf{F}_{2\to 1}$.

Theorem 4.7.

$$\mathbf{B} = \boldsymbol{\nabla}\times\left(\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}_q)}{\|\mathbf{r}_p - \mathbf{r}_q\|}\,\mathrm{d}v_q\right). \tag{33}$$

Theorem 4.8.

$$\boxed{\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}.} \tag{34}$$

Corollary 4.9.

$$\oint_{\partial\Sigma} \langle \mathbf{B}, \mathrm{d}\mathbf{r}\rangle_I = \mu_0\int_{\Sigma} \langle \mathbf{J}, \mathrm{d}\mathbf{s}\rangle_I \tag{35}$$

Theorem 4.10.

$$\operatorname{div} \mathbf{B} = 0. \tag{36}$$

Definition 4.4.

$$\mathbf{A}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_V \mathbf{J}(\mathbf{r}_q) \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \, \mathrm{d}v_q\,, \qquad [A] = \mathrm{T\,m}. \tag{37}$$

Proposition 4.11. *The vector potential is not unique,* $\mathbf{A}$ *and* $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\xi$ *are equivalent.*

Proposition 4.12.

$$\phi_m = \int\limits_{\partial\Sigma} \langle \mathbf{A}, \mathrm{d}\mathbf{r} \rangle_I. \tag{38}$$

Theorem 4.13. *If* $\mathbf{J} = \mathbf{0}$*, then*

$$\nabla^2 \Phi_m = 0 \tag{39}$$

5 Magnetostatics in matter

Definition 5.1. *Dipolar magnetic moment* or *magnetic moment.*

$$\mathbf{m} \coloneqq \frac{1}{2} \int \mathbf{r}_q \times \mathbf{J}(\mathbf{r}_q) \, \mathrm{d}v_q\,, \qquad [m] = \mathrm{A\,m^2} = \mathrm{J\,T^{-1}}. \tag{40}$$

Proposition 5.1. *For a wire,*

$$\mathbf{m} = \frac{1}{2} I \oint\limits_\gamma \mathbf{r} \times \mathrm{d}\mathbf{r}\,. \tag{41}$$

Proposition 5.2. *Let* γ *be a curve such that* Γ *is contained in a plane. Then,* $\mathbf{m} = IS\mathbf{n}$*.*

Theorem 5.3. *If* ρ_m, ρ_q *are constants,* $\mathbf{m} = \frac{q}{2m}\mathbf{L}$*.*

Proposition 5.4. *If* $\|\mathbf{r}\| \gg 1$*, then*

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{\|\mathbf{r}\|^3}, \tag{42}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[\frac{3\langle \mathbf{m}, \mathbf{r} \rangle_I \mathbf{r}}{\|\mathbf{r}\|^5} - \frac{m}{\|\mathbf{r}\|^3} \right] = -\mu_0 \boldsymbol{\nabla}\Phi_m, \;\; \Phi_m = \frac{1}{4\pi} \frac{\langle \mathbf{m}, \mathbf{r} \rangle_I}{\|\mathbf{r}\|^3} \tag{43}$$

Definition 5.2. The north pole of a magnet is where the magnetic field lines leave the magnet, which is equivalent to the extreme that points to the geographic north pole.

Theorem 5.5. *Magnetic force and torque on a dipole*

$$\mathbf{F} = \boldsymbol{\nabla}\langle \mathbf{m}, \mathbf{B} \rangle_I, \qquad \boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \tag{44}$$

Definition 5.3. *Magnetization*

$$\mathbf{M} \coloneqq \frac{1}{\Delta V} \sum_{i=1}^{n} \mathbf{m}_i, \; \frac{1}{\Delta V} \frac{1}{2} \int\limits_{\Delta V} \mathbf{r} \times \mathbf{J} \, \mathrm{d}v\,, \qquad [M] = \mathrm{A\,m^{-1}}. \tag{45}$$

Theorem 5.6. *At the exterior (there is no* $\mathbf{J}$*).*

$$\mathbf{A}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_V \frac{\mathbf{J}_M(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|} \, \mathrm{d}v_q + \frac{\mu_0}{4\pi} \oint\limits_{\partial V} \frac{\mathbf{K}_M(\mathbf{r}_p)}{\|\mathbf{r}_p - \mathbf{r}_q\|} \, \mathrm{d}s_q\,, \tag{46}$$

where $\mathbf{J}_m \coloneqq \operatorname{curl} \mathbf{M}, \mathbf{K}_m \coloneqq \mathbf{M} \times \mathbf{n}$*.*

Theorem 5.7. *At the exterior,*

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_V [\mathbf{J}_f(\mathbf{r}_q) + \mathbf{J}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}v_q\,, \tag{47}$$

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_{\operatorname{int} V} \rho_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}v_q + \frac{\mu_0}{4\pi} \oint\limits_{\partial V} \sigma_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}s \tag{48}$$

where $\rho_m \coloneqq -\operatorname{div} \mathbf{M}, \sigma_m \coloneqq \langle \mathbf{M}, \mathbf{n} \rangle_I$*.*

Theorem 5.8. *In general,*

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_{\operatorname{int} V} [\mathbf{J}_f(\mathbf{r}_q) + \mathbf{J}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}v_q \tag{49}$$

$$+ \frac{\mu_0}{4\pi} \oint\limits_{\partial V} [\mathbf{K}_f(\mathbf{r}_q) + \mathbf{K}_m(\mathbf{r}_q)] \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}s_q\,. \tag{50}$$

$$\mathbf{B}(\mathbf{r}_p) = \frac{\mu_0}{4\pi} \int\limits_{\operatorname{int} V} \rho_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}v_q + \frac{\mu_0}{4\pi} \oint\limits_{\partial V} \sigma_m(\mathbf{r}_q) \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}s \tag{51}$$

$$+ \frac{\mu_0}{4\pi} \int\limits_{\operatorname{int} V} \mathbf{J}_f(\mathbf{r}_q) \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}v_q + \frac{\mu_0}{4\pi} \oint\limits_{\partial V} \mathbf{K}_f(\mathbf{r}_q) \times \frac{\mathbf{r}_p - \mathbf{r}_q}{\|\mathbf{r}_p - \mathbf{r}_q\|^3} \, \mathrm{d}s_q \tag{52}$$

Definition 5.4. Magnetic intensity

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \qquad [\mathbf{H}] = \mathrm{A\,m^{-1}}. \tag{53}$$

Theorem 5.9. *Ampere's theorem for magnetic intensity*

$$\operatorname{curl} \mathbf{H} = \mathbf{J}_f \Rightarrow \oint\limits_{\partial\Sigma} \langle \mathbf{H}, \mathrm{d}\mathbf{r} \rangle_I = \int\limits_\Sigma \langle \mathbf{J}_f, \mathrm{d}\mathbf{s} \rangle_I \tag{54}$$

Definition 5.5. *Magnetic poles density,*

$$\rho_m \coloneqq \operatorname{div} \mathbf{H}, \qquad \sigma_m \coloneqq \langle \mathbf{M}, \mathbf{n} \rangle_I. \tag{55}$$

Definition 5.6. *Magnetic susceptibility,*

$$\mathbf{M} = \chi_m \mathbf{H}, \qquad [\chi_m] = 1. \tag{56}$$

6 Slow variable fields

Axiom 6 (Lorentz force for non-stationary conditions)**.**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{57}$$

Definition 6.1. Emf,

$$\mathcal{E} \coloneqq \oint\limits_\gamma \langle \mathbf{E}_{\mathrm{ef}}, \mathrm{d}\mathbf{r} \rangle_I. \tag{58}$$

Proposition 6.1. *For open circuits, $\mathcal{E} = \Delta\Psi$.*

Axiom 7 (Faraday's Law)**.**

$$\mathcal{E} = \frac{\mathrm{d}\phi}{\mathrm{d}t} \Leftrightarrow \oint\limits_{\partial\Sigma} \langle \mathbf{E}, \mathrm{d}\mathbf{r} \rangle_I = -\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Sigma} \langle \mathbf{B}, \mathrm{d}\mathbf{s} \rangle_I \tag{59}$$

Theorem 6.2. *Maxwell-Faraday law for stationary systems (general, by contrast to the original Faraday's law)*

$$\operatorname{curl} \mathbf{E} = -\frac{\partial B}{\partial t}. \tag{60}$$

Theorem 6.3. *Faraday's law for moving systems.*

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{61}$$

Definition 6.2. *Mutual inductance* or *induction coefficient.*

$$M_{2\leftarrow 1} \coloneqq \frac{\mu_0}{4\pi} \oint\limits_{\gamma_2} \oint\limits_{\gamma_1} \frac{1}{\|\mathbf{r}_2 - \mathbf{r}_1\|} \langle \mathrm{d}\mathbf{r}_2 \,, \mathrm{d}\mathbf{r}_1 \rangle_I. \tag{62}$$

Corollary 6.4. $M_{2\leftarrow 1} = M_{1\leftarrow 2}$.

Proposition 6.5. $\phi_{2\leftarrow 1} = M_{2\leftarrow 1} I_1$.

Corollary 6.6. $\mathcal{E}_2 = -\dot{\phi_m} = -M_{2\leftarrow 1}\dot{I}_1$.

Definition 6.3.

$$L \coloneqq \frac{\mu_0}{4\pi} \oint\limits_{\gamma} \oint\limits_{\gamma} \frac{1}{\|\mathbf{r}_2 - \mathbf{r}_1\|} \langle \mathrm{d}\mathbf{r}_2 \,, \mathrm{d}\mathbf{r}_1 \rangle_I. \tag{63}$$

Proposition 6.7. *Equations for a transformer (the last one is the coupling coefficient).*

$$\frac{n_1}{n_2} = \frac{\phi_1}{\phi_{2\leftarrow 1}} = \frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{L_1}{M_{2\leftarrow 1}} = \frac{M_{1\leftarrow 2}}{L_2} = \sqrt{\frac{L_1}{L_2}}, \tag{64}$$

$$L_1 L_2 = M_{2\rightarrow 1}^2, \qquad k \coloneqq \frac{M_{2\leftarrow 1}}{\sqrt{L_1 L_2}}. \tag{65}$$

Theorem 6.8. *For rigid and stationary circuits without hysteresis, $W_m = I\phi_m$.*

Theorem 6.9. *Stationary circuits and without hysteresis,*

$$W_m = \frac{1}{2}\sum_{i=1}^{n} I_i \phi_i = \frac{1}{2}\sum_{i=1}^{n} I_i \oint\limits_{\gamma_i} \langle \mathbf{A}_i, \mathrm{d}\mathbf{r} \rangle_I = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} I_i I_j. \tag{66}$$

Corollary 6.10. *Particular systems.*

1. *Coil/bobbin: $W_m = \frac{1}{2} L I^2$*
2. *Two coupled circuits: $\frac{1}{2} L_1 I_1^2 + M I_1 I_2 + \frac{1}{2} L_2 I_2^2$*

Theorem 6.11. *For static circuits*

$$W_m = \frac{1}{2} \int\limits_{\mathbb{R}^3} \langle \mathbf{H}, \mathbf{B} \rangle_I \, \mathrm{d}v = W_b. \tag{67}$$

Theorem 6.12. *With and without battery respectively*

$$\mathbf{F} = \boldsymbol{\nabla} W_m|_I \,, \qquad \mathbf{F} = \boldsymbol{\nabla} W_m|_\Phi \,. \tag{68}$$

Definition 6.4. *Magnetic reluctance*

$$\mathcal{R} \coloneqq \oint\limits_{\gamma} \frac{1}{\mu S} \, \mathrm{d}r \,. \tag{69}$$

Proposition 6.13. *$nI = \phi_m \mathcal{R}$.*

Proposition 6.14 (Langevin diamagnetism)**.** *Let S be a system of N atoms per unit volume separated by a mean square distance $\langle r^2 \rangle_I$. If the atoms have Z orbiting electrons of mass m_e and charge q, then*

$$\chi_e = -\frac{q^2 Z}{6 m_e} \mu_0 N \langle r^2 \rangle_I. \tag{70}$$

Proposition 6.15. *Formulas for cylindrical coil*

$$L = \frac{\mu_0 N^2 \pi r^2}{l} \tag{71}$$

Proposition 6.16. *Formulas for toroidal coil*

$$L = \mu_0 N^2 \left(b - \sqrt{b^2 - a^2} \right), \qquad W_m = \frac{\mu_0 N^2 I^2}{2} \left(b - \sqrt{b^2 - a^2} \right). \tag{72}$$

7 Maxwell equations

Definition 7.1. Displacement current

$$\mathbf{J}_d \coloneqq \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{73}$$

Proposition 7.1. *Lorentz force*

$$\mathbf{F} = \int\limits_{\Omega} \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \, \mathrm{d}v \,. \tag{74}$$

Theorem 7.2 (Maxwell's equations in vacuum)**.**

$$\operatorname{div} \mathbf{E} = \frac{\rho_t}{\epsilon_0} \tag{75}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{76}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{77}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}_t + \mu_0 \mathbf{J}_d, \tag{78}$$

in integral form,

$$\oint\limits_{\partial\Omega} \langle \mathbf{E}, \mathrm{d}\mathbf{s} \rangle_I = \frac{1}{\epsilon_0} \int\limits_{\Omega} \rho_t \, \mathrm{d}v \tag{79}$$

$$\oint\limits_{\partial\Omega} \langle \mathbf{B}, \mathrm{d}\mathbf{s} \rangle_I = 0 \tag{80}$$

$$\oint\limits_{\partial\Sigma} \langle \mathbf{E}, \mathrm{d}\mathbf{r} \rangle_I = -\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Sigma} \langle \mathbf{B}, \mathrm{d}\mathbf{s} \rangle_I \tag{81}$$

$$\oint\limits_{\partial\Sigma} \langle \mathbf{B}, \mathrm{d}\mathbf{r} \rangle_I = \mu_0 \int\limits_{\Sigma} \langle \mathbf{J}_t, \mathrm{d}\mathbf{s} \rangle_I + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Sigma} \langle \mathbf{E}, \mathrm{d}\mathbf{s} \rangle_I. \tag{82}$$

Theorem 7.3 (Maxwell's equations in media)**.**

$$\operatorname{div}\mathbf{D} = \rho_f \tag{83}$$
$$\operatorname{div}\mathbf{B} = 0 \tag{84}$$
$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{85}$$
$$\operatorname{curl}\mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{86}$$

in integral form,

$$\oint_{\partial\Omega} \langle \mathbf{D}, \mathrm{d}\mathbf{s}\rangle_I = \int_{\Omega} \rho_f \,\mathrm{d}v \tag{87}$$
$$\oint_{\partial\Omega} \langle \mathbf{B}, \mathrm{d}\mathbf{s}\rangle_I = 0 \tag{88}$$
$$\oint_{\partial\Sigma} \langle \mathbf{E}, \mathrm{d}\mathbf{r}\rangle_I = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \langle \mathbf{B}, \mathrm{d}\mathbf{s}\rangle_I \tag{89}$$
$$\oint_{\partial\Sigma} \langle \mathbf{H}, \mathrm{d}\mathbf{r}\rangle_I = \int_{\Sigma} \langle \mathbf{J}_f, \mathrm{d}\mathbf{s}\rangle_I + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \langle \mathbf{D}, \mathrm{d}\mathbf{s}\rangle_I \tag{90}$$

Definition 7.2.

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{91}$$

Proposition 7.4.

$$\operatorname{div}\mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0. \tag{92}$$

Theorem 7.5. *Electromagnetic energy*

$$W_{em} = \frac{1}{2}\int_{\Omega} \langle \mathbf{H}, \mathbf{B}\rangle_I + \langle \mathbf{D}, \mathbf{E}\rangle_I \,\mathrm{d}v\,. \tag{93}$$

Definition 7.3. *Poynting vector*, $\mathbf{S} = \mathbf{E}\times\mathbf{H}$.

Theorem 7.6.

$$\frac{\mathrm{d}W_k}{\mathrm{d}t} + \frac{\mathrm{d}W_{em}}{\mathrm{d}t} = -\int_{\Sigma} \langle \mathbf{S}, \mathrm{d}\mathbf{s}\rangle_I, \tag{94}$$

$$\int_{\Omega} \langle \mathbf{J}, \mathbf{E}\rangle_I \,\mathrm{d}v + \frac{\partial}{\partial t}\int_{\Omega} \frac{\langle \mathbf{H}, \mathbf{B}\rangle_I + \langle \mathbf{D}, \mathbf{E}\rangle_I}{2}\,\mathrm{d}v = -\oint_{\partial\Omega} \langle \mathbf{E}\times\mathbf{H}, \mathrm{d}\mathbf{s}\rangle_I \tag{95}$$

Corollary 7.7.

$$\eta = \frac{\mathrm{d}W}{\mathrm{d}V} = \frac{\langle \mathbf{H}, \mathbf{B}\rangle_I + \langle \mathbf{D}, \mathbf{E}\rangle_I}{2}. \tag{96}$$

Definition 7.4. Maxwell stress tensor

$$\sigma_{ij} := \epsilon\left(E_iE_j - \frac{\delta_{ij}}{2}E^2\right) + \frac{1}{\mu}\left(B_iB_j - \frac{\delta_{ij}}{2}B^2\right), \tag{97}$$

$$\boldsymbol{\sigma} = \epsilon\left(\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}\mathbb{I}\right) + \frac{1}{\mu}\left(\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}\mathbb{I}\right). \tag{98}$$

Theorem 7.8. *Electromagnetic force per unit volume*

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{f} + \epsilon\mu\frac{\partial \mathbf{S}}{\partial t}. \tag{99}$$

Definition 7.5. Electromagnetic impulse

$$\mathbf{P}_{em} := \frac{1}{u^2}\int_{\Omega} \mathbf{S}\,\mathrm{d}v\,. \tag{100}$$

Theorem 7.9. *Boundary conditions*

$$\langle \mathbf{n}_{1\to 2}, \mathbf{B}_2 - \mathbf{B}_1\rangle_I = 0 \tag{101}$$
$$\mathbf{n}_{1\to 2}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mathbf{K}_f + \mathbf{K}_m \tag{102}$$
$$\langle \mathbf{n}_{1\to 2}, \mathbf{H}_2 - \mathbf{H}_1\rangle_I = \sigma_m \tag{103}$$
$$\mathbf{n}_{1\to 2}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f \tag{104}$$
$$\langle \mathbf{n}_{1\to 2}, \mathbf{D}_2 - \mathbf{D}_1\rangle_I = \sigma_f \tag{105}$$
$$\mathbf{n}_{1\to 2}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{106}$$

Proposition 7.10. *There exists functions* $\Phi, \mathbf{A}$ *such that*

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \qquad \mathbf{E} = -\operatorname{grad}\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{107}$$

Definition 7.6. Gauge transformations

$$\Phi' = \Phi - \frac{\partial \xi}{\partial t}, \qquad \mathbf{A}' = \mathbf{A} + \operatorname{grad}\xi. \tag{108}$$

Definition 7.7. Coulomb and Lorenz conditions

$$\nabla^2\xi = -\operatorname{div}\mathbf{A} \Rightarrow \operatorname{div}\mathbf{A}' = 0, \qquad \nabla^2\xi - \epsilon\mu\frac{\partial \xi}{\partial t} = 0 \Rightarrow \operatorname{div}\mathbf{A}' + \epsilon\mu\frac{\partial \Phi'}{\partial t} \tag{109}$$

Proposition 7.11. *In general,*

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\operatorname{div}\mathbf{A}) = -\frac{\rho_t}{\epsilon}, \qquad \nabla^2\mathbf{A} - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left[\operatorname{div}\mathbf{A} + \epsilon\mu\frac{\partial \Phi}{\partial t}\right] = \tag{110}$$

Theorem 7.12. *If Coulomb conditions,*

$$\nabla^2\Phi = -\frac{\rho}{\epsilon}, \qquad \nabla^2\mathbf{A} - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} - \epsilon\mu\frac{\partial}{\partial t}(\boldsymbol{\nabla}\Phi) = -\mu\mathbf{J}_t \tag{111}$$

and search solutions.

Theorem 7.13. *If Lorenz condition,*

$$\nabla^2\Phi - \epsilon\mu\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho_t}{\epsilon}, \qquad \nabla^2\mathbf{A} - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}_t, \tag{112}$$

which have the solutions

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon}\int_{\Omega} \rho(\mathbf{r}_q, t_0)\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|}\,\mathrm{d}v_q + \Phi_0, \tag{113}$$

$$\mathbf{A}(\mathbf{r}_p, t) = \frac{\mu}{4\pi}\int_{\Omega} \mathbf{J}(\mathbf{r}_q, t_0)\frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|}\,\mathrm{d}v_q + \mathbf{A}_0, \tag{114}$$

where

$$t_0 := t - \frac{\|\mathbf{r}_p - \mathbf{r}_q\|}{u}, \qquad u := \frac{1}{\sqrt{\epsilon\mu}}, \qquad c := \frac{1}{\sqrt{\epsilon_0\mu_0}}. \tag{115}$$

Corollary 7.14. *Single particle, Lienard-Wiechert potentials ($\mathbf{n} = (\mathbf{r}_p - \mathbf{r}_q)/\|\mathbf{r}_p - \mathbf{r}_q\|$)*

$$\Phi(\mathbf{r}_p, t) = \frac{1}{4\pi\epsilon} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \left. \frac{1}{1 - \langle \mathbf{v}, \mathbf{n} \rangle_I / u} \right|_{t_0}, \qquad \mathbf{A}(\mathbf{r}_p, t) = \frac{\mu}{4\pi} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|} \left. \frac{q\mathbf{v}}{1 - \langle \mathbf{v}, \mathbf{n} \rangle_I / u} \right|_{t_0}, \tag{116}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{1}{\|\mathbf{r}_p - \mathbf{r}_q\|^2} \frac{(1-\beta^2)(\mathbf{n} - \boldsymbol{\beta})}{\left(1 - \langle \boldsymbol{\beta}, \mathbf{n} \rangle_I\right)^3} + \frac{q}{4\pi\epsilon_0} \frac{1}{c\|\mathbf{r}_p - \mathbf{r}_q\|} \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{\left(1 - \langle \boldsymbol{\beta}, \mathbf{n} \rangle_I\right)^3}, \tag{117}$$

$$\mathbf{B} = \frac{1}{c} \mathbf{n} \times \mathbf{E}. \tag{118}$$

8 Electromagnetic potentials

Theorem 8.1. *Wave equations*

$$\nabla^2 \mathbf{E} - \mu g \frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{119}$$

$$\nabla^2 \mathbf{H} - \mu g \frac{\partial \mathbf{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \tag{120}$$