1 Introduction

Definition 1.1. Let $y(x)$ be an unknown function. Then, a *differential equation* is an equation where x, $y(x)$, and its derivatives intervene. We write this equation in the following ways.

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0 \qquad F\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) \tag{1}$$

Definition 1.2. Let $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ be a differential equation. Then, we define the *order of the equation* as the order of the maximum derivative in the equation.

Definition 1.3. Let $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ be a differential equation. Then, we define the *degree of the equation* as the number of the exponent in the highest order derivative.

Definition 1.4. If there are more than one differential equations, we say that we have a *system of differential equations.*

Definition 1.5. If the function y that intervenes in the differential equation has more than one variable, we say that we have a *partial differential equation*, a *differential equation in partial derivative*, or simply *PDE*. If not, we say we have a *ordinary differential equation*, a *differential equation in ordinary derivative*, or simply *ODE*.

Definition 1.6. Let $F = 0$ be a differential equation. We define *finding the solution to the differential equation* as finding the function y that satisfies this equation.

Definition 1.7. In a differential equation, we define the *general solution* as the set of all the solutions to the equation.

Definition 1.8. In a differential equation, we define a *particular solution* as an element of the general solution.

Proposition 1.1. *Let be a differential equation of the form $f(x)dx = g(y)dy$. Then, a function $y(x)$ is a solution of the differential equation if and only if it is a solution for*

$$\int f(x)dx = \int g(y)dy \tag{2}$$

.

Proposition 1.2. *Let be a differential equation of the form $f(x)dx = g(y)dy$, with the initial condition $y(a) = b$. Then, a function $y(x)$ is a solution of the differential equation and the satisfies the initial condition if and only if it satisfies*

$$\int_a^x f(x)dx = \int_b^y g(y)dy. \tag{3}$$

Definition 1.9. Let $f(x, y) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say *the function satisfies the Lipschitz condition with respect to the variable y* if it is true that

$$\forall (x, y_1), (x, y_2)\ \exists k \mid |f(x, y_1) - f(x, y_2)| \leq k\, |y_1 - y_2| \tag{4}$$

And we call the constant k as *the Lipschitz constant of the function* [ ].

Proposition 1.3. *Every function such that $\frac{df}{dy} \neq \infty$ in S satisfies Lipschitz Condition. It's enough to take $A \geq \max_S |\frac{\partial f}{\partial y}|$. Furthermost, in most cases the theorem fulfills $\forall x$, given that there's no limit for h, h'.*

Proposition 1.4. *If $\exists!$ a solution such that $y = b$ when $x = a$, the general solution to the first order differential equation is a function $y(x)$ with one trivial constant: the value of y when $x = a$, i.e. b*

Theorem 1.5. *Let $S \subset \mathbb{R}^2$ be a closed set defined by $|x - a| \leq h$ and $|y - b| \leq k$, and let $f(x, y,)$ a continuous function in S which satisfies Lipschitz Condition and which is bounded by real number M. Then, the differential equation $y' = f(x, y)$ has a unique solution such that $y = b$ when $x = a$, in the interval $|x - a| \leq h'$, where $h' = \min(h, k/M)$*

Theorem 1.6. *If a system of n first order differential equations $y_1' = f_1(x, y_1, y_n), y_2' = f_2(x, y_1, y_n) \dots y_n' = f_n(x, y_1, y_n)$, the functions $f_1 \dots f_n$ are continuous in the S region defined by $|x - a| \leq h$, $|y_1 - b_1| \leq k_1 \dots |y_n - b_n| \leq k_n$, and if the satisfy (in S) Lipschitz Condition $|f_i(x, y_1 \dots y_n) - f_i(x, z_1, z_n)| \leq A_1|y_1 - z_1| + \dots + A_n|y_n - z_n|$ then in the I interval, $|x - a| \leq h' = min(h, \frac{k_i}{M})$ where $M \geq |f_i(x, y_1 \dots y_n)$, there exists a unique set of continuous functions $y_1(x) \dots y_n(x)$ with continuous derivatives inside I, which are solution to the system of differential equations and satisfy $y_i(a) = b_i$*

Proposition 1.7. *Given that a differential equation of n order in y, $\frac{d^ny}{dx^n} = f(x, y, \frac{dy}{dx} \dots \frac{d^{n-1}y}{dx^{n-1}})$ is equivalent to n first order differential equations in $y, y_1 \dots y_{n-1}$ which are $\frac{dy}{dx} = y_1$, $\frac{dy_1}{dx} = y_2 \dots \frac{dy_{n-2}}{dx} = y_{n-1}$, $\frac{dy_{n-1}}{dx} = f(x, y, y_1 \dots y_{n-1})$, a differential equation of order n has a unique solution with $y(x_0) = b_0, y'(x_0) = b_1 \dots y^{(n-1)}(x_0) = b_{n-1}$ if the previous conditions are fulfilled.*

2 First order equations

Definition 2.1. Every family is expressed by a relation $F(x, y, C)$ and they're the geometric representation of a differential equation's general solution.

Definition 2.2. Given a family of curves $F(x, y, C) = 0$ we find out its differential equation $f(x, y, dy/dx) = 0$. The differential equation of the trajectories will be $f(x, y, -dx/dy) = 0$.

Definition 2.3. The Clairaut's Equation is defined by

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

Proposition 2.1. *The family of non-parallel lines $y = Cx + f(C)$ represents the general solution of the Clairaut's Equation.*

Definition 2.4. A singular solution is an additional solution not included in the general one (it doesn't come up with any C value).

Definition 2.5. An envelope is a curve which is tangent in every of its points to the general solution $y = Cx + f(C)$

Proposition 2.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and let y be another function of the form $y(x) = cx + f(c)$. Then, the envelope satisfies that*

$$x = -\frac{\partial f}{\partial c} \tag{5}$$

Proposition 2.3. *If the envelope exists, it must satisfy:*

$$y = Cx + f(C)$$
$$x = -\frac{\partial f}{\partial C}$$

Definition 2.6. A linear differential equation is an equation such that:

$$A(x)\frac{dy}{dx} + B(x)y + C(x) = 0 \Leftrightarrow \frac{dy}{dx} + P(x)y = Q(x)$$

Proposition 2.4. *The general solution to a linear differential equation is*

$$y = C\exp\left\{\left[-\int P(x)dx\right]\right\} + \exp\left\{\left[-\int P(x)dx\right]\right\}\int Q(x)\exp\left\{\left[\int P(x)dx\right]\right\}dx$$

Definition 2.7. A linear differential is defined as reduced if $Q(x) = 0$

$$\frac{dy}{dx} + P(x)y = 0$$

Proposition 2.5. *The general solution to a reduced linear differential equation is*

$$y = D\exp\left\{[-\int P(x)dx]\right\}$$

Proposition 2.6. *Let y_1 be a particular solution to the reduced differential equation, then the general soluton is $y = Cy_1$*

Proposition 2.7. *Let y_1 be a particular solution to a reduced differential equation and y_2 to the complete differential equation, then the general solution to the complete differential equation is $y = Cy_1 + y_2$*

Definition 2.8. The Bernoulli's equation is defined as:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$\forall n \in \mathbb{R}$, being linear for $n = 0, n = 1$

Proposition 2.8. *In the rest of cases, it can be reduced to a linear differential equation using the substitution $z = y^{1-n}$*

Definition 2.9. The Riccati's equation is defined as:

$$y' = P(x)y^2 + Q(x)y + R(x)$$

Generally, it can't be solved, but given a particular solution we can find the general one.

Proposition 2.9. *Given a particular solution*

$$y_p = P(x)y_p^2 + Q(X)y_P + R(x)$$

We can transform Riccati's equation into Bernoulli's equation with $n = 2$, realizing a change of variables we eventually obtain a linear equation.

Proposition 2.10. *Given two particular solutions to a Riccati's equation we can find the general solution.*

Definition 2.10. $F(x, y)$ is an nth-grade homogeneous equation if and only if

$$F(tx, ty) \equiv t^n F(x, y) \quad \forall t \in \mathbb{R}$$

Proposition 2.11. *If $F(x, y)$ is an homogeneous equation of grade n, $G(x, y)$ m-th grade homogeneous, then $FG, F/G$ are homogeneous equations of grade $n + m$, $n - m$, respectively.*

Proposition 2.12. *If $F(x, y)$ is an homogeneous equation of grade 0, then $F(x, y)$ is only a function of y/x*

Definition 2.11. An homogeneous differential equation is one such that

$$M(x, y)dx + N(x, y)dy = 0$$

Being M, N homogeneous functions both with same grade

Proposition 2.13. *An homogeneous differential equation can be resolved with an easy change of variables rigged by $y = vx$ or $x = yv$, depending on whether we want the solution to be a function of x or a function of y $(x(y), y(x))$*

Definition 2.12. An homothecy is a linear transformation of a point x, y to a point kx, ky

Proposition 2.14. *The curves representing solutions can be transform among themselves between homothecy*

Definition 2.13. An exact differential equation is an homogeneous differential equation such that

$$M(x, y)dx + N(x, y)dy = 0 \text{ exact } \Leftrightarrow \exists u(x, y) \| du \equiv M(x, y)dx + N(x$$

Proposition 2.15. *Exact differential equations can be solved using*

$$du = 0 \Rightarrow u(x,y) = C$$

Proposition 2.16.

$$M(x,y)dx + N(x,y)dy = 0 \quad exact \quad \Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Definition 2.14. A non-null function $\mu(x,y)$ is an integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$ if and only if

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is an exact differential equation

Proposition 2.17. *There $\exists$ ∞ integrating factors.*

Proposition 2.18. *Integrating factors depending only on x can be found*

$$\mu(x) = \exp\left\{\left[\int \frac{M_y - N_x}{N} dx\right]\right\}$$

Proposition 2.19. *Integrating factors depending only on x can be found*

$$\mu(x) = \exp\left\{\left[\int \frac{N_x - M_y}{M} dx\right]\right\}$$

Proposition 2.20. *Let $\vec{V} = (V_x, V_y, 0)$ be a vector on a plane, such that*

$$V_x = M(x,y), \ V_y = N(x,y)$$

Then

$$\int \vec{V}\cdot d\vec{l} \ \ \textit{independant of the path} \Leftrightarrow \nabla\times\vec{V} = 0 \Leftrightarrow (\nabla\times\vec{V})_z = 0 \Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Leftrightarrow Mdx + Ndy = 0 \ \ \textit{exact}$$

3 Linear equations

Definition 3.1. A nth-order linear differential equation is:

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = R(x)$$

Proposition 3.1. *Let a solution to the reduced equation be cancelled, as they ordinary derivatives until $(n-1)$ in some x_0 point, then the solution is $y(x) \equiv 0$*

Proposition 3.2. *Let $u_1 \dots u_k$ be solutions to the reduced diff. eq., then a lineal combinations of those ones $c_1u_1 + \dots c_ku_k$ are also solution.*

Proposition 3.3. *Let y_1 and y_2 be solutions to the complete equation, $y_1 - y_2$ is solution to the reduced one.*

Proposition 3.4. *Let y_1 be solution to the complete equation and u_1 to the reduced one, then $y_1 + u_1$ is solution to the complete one.*

Theorem 3.5. *The general solution to a complete differential equation can be obtained adding the general solution of the reduced one to the particular solution of the complete one.*

Definition 3.2. The wronskian of n functions is $W(u_1 \dots u_n)$ defined as

$$W(u_1 \dots u_n) = \begin{vmatrix} u_1 & \dots & u_n \\ u_1' & \dots & u_n' \\ \vdots & & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix}$$

$u_1 \dots u_n$ linearly dependant $\Leftrightarrow \exists c_1 \dots c_n$ not-null so that $c_1u_1 + \cdots + c_nu_n \equiv 0$

Theorem 3.6. *Let be n linearly dependant functions and exists their derivatives until $(n-1)$, then its wronskian $\equiv 0$*

Theorem 3.7. *Let the wronskian of n solutions to a reduced differential equation get cancelled in a point, then the solutions are linearly dependant and its wronskian gets cancelled $\forall$ points.*

Theorem 3.8. *Every solution to the reduced equation can be expressed as a linear combination of n linearly independent solutions*

Definition 3.3. A reduced second order differential equation with constant coefficients is one such

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

Definition 3.4. The auxiliary equation is the reduced equation written as a polynomial.

$$m^2 + pm + q = 0$$

Definition 3.5. A reduced nth-order differential equation with constant coefficients is one such

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_ny = 0$$

4 Laplace transform

Proposition 4.1. *Let $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$*

Proposition 4.2. *Let $\mathcal{L}\{f_1(t)\} = F_1(s)$, $\mathcal{L}\{f_2(t)\} = F_2(s)$, then $\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1F_1(s) + c_2F_2(s)$*

Proposition 4.3. *Let $F(s) = s_0$ converge for $s = s_0$, then it converges $\forall s > s_0$*

Proposition 4.4. *For any continuous function $f(t)$ in the interval $0 \leq t \leq \infty$ (or at least discontinuous in a finite number of points) which satisfies $\mid f(t) \mid Me^{at}$, there exists the Laplace Transform $\forall s > a$.*

Proposition 4.5. *The Laplace Transform the nth-derivative of a function is*

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k}f^{(k)(0)}$$

Proposition 4.6. *$\forall \varepsilon > 0$ arbitrarily small, if $u(x)$ is a continuous function, we can find a polynomial $P(x)$ $\mid |u(x) - P(x)| \leq \varepsilon, \forall x$.*

Lemma 4.7. *If $u(x)$ is continuous in $[0,1]$ and $\int_0^1 x^n u(x)dx = 0$ $\forall n \in \mathbb{N}*$, then $u(x) = 0$ in $[0,1]$.*

Theorem 4.8. *The Laplace Transform $F(s)$ of a continuous function $f(t)$ isn't the Transform of any other continuous function.*

Proposition 4.9. *The Laplace Transform of $t^nf(t), \forall n \in \mathbb{N}$ is $(-1)^nF^{(n)}(s)$*

Proposition 4.10. *The Laplace Transform of $\int_0^t f(\tau)d\tau$ is $F(s)/s$*

Theorem 4.11. *Let $F(s)$ and $G(s)$ be the Laplace Transforms of $f(t)$ and $g(t)$, then then product $F(s)G(s)$ is the Laplace Transform of $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$. Both integrals are named the convolution of $f(t), g(t)$.*

Proposition 4.12. *The Laplace Transform of $\mathcal{L}\{f(t-a)\theta(t-a)\}$, where $\theta(t)$ is the Heaviside step function, is $e^{-as}\mathcal{L}\{f(t)\}$*

5 Power series

Definition 5.1. Let $f(x)$ be a function, then $f(x)$ is *analytical* in a if it is an infinitely differentiable function such that the Taylor series at a converges pointwise to $f(x)$ for a in a neighbourhood of x.

Definition 5.2. In equations of the form (5.1), a is an *ordinary point* if $P(x)$, $Q(x)$ are analytical in a.

Definition 5.3. If one the functions isn't analytical in a, but both of them are analytical in a neighbourhood of a, then a is a *singular point*.

Definition 5.4. Consider the previous (5.1) equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

a singular point a is *regular singular* if $(x-a)P(x)$, $(x-a)^2Q(x)$ are analytical, thus they allow Taylor Series expansion

Definition 5.5. The indicial equation in a series expansion with Frobenius' Method is

$$r(r-1) + rp_0 + q_0 = F(r) = 0$$

Being p_0 and q_0 the zero-order terms in Taylor expansions of $p(x)$ and $q(x)$.

Definition 5.6. The hypergeometric differential equation, also called Gauss' equation is

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0 \qquad (6)$$

which has 2 regular singular points ($x = 0$ and $x = 1$), the rest of them are just ordinary points.

Proposition 5.1. *If r_2 isn't a positive integer, namely if γ isn't zero or a negative integer, there exists a first solution*

$$y_1 = \sum_{n=0}^{\infty} c_nx^n = 1 + c_1x + c_2x^2 + c_3x^3 + \dots \quad c_0 = 1$$

Definition 5.7. The first solution around the origin of the hypergeometric differential equation is

$$y_1 = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!} = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha}{3!\gamma(}$$

named the *hypergeometric series* $F(\alpha,\beta,\gamma,x)$ which converges $\forall |x| < 1$. Where $(q)_n = q^{\overline{n}}$ is the rising Pochhammer symbol or commonly named the rising factorial.

Proposition 5.2. *If r_2 isn't a positive integer, namely if γ isn't zero or a negative integer, there exists a second solution*

$$y_2 = x^{1-\gamma}y_1 = \sum_{n=0}^{\infty} c_nx^{n+1-\gamma} = x^{1-\gamma}(1 + c_1x + c_2x^2 + c_3x^3 + \dots) \quad c_0 =$$

Definition 5.8. Legendre's differential equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (7)$$

with n not necessarily being an integer, $n \in \mathbb{R}$

Definition 5.9. If n is zero or a positive integer, the analytical series solution to (5.4) is finite. Said series is named *Legendre polynomial* of nth-order.

$$P_n(x) = F(n+1, -n, 1, \frac{1}{2}(1-x)) = \sum_{n=0}^{\infty} \frac{(n+1)_n(n)_n}{(1)_n}\frac{((x-1)/2)^n}{n!}$$

where $(q)_n = q^{\overline{n}}$ is the rising factorial.

Proposition 5.3. $P_n(x) = (-1)^nP_n(-x)$

Proposition 5.4. *Orthogonality of Legendre's Polynomials. If $m \neq n$, then $\forall x \in (-1, 1)$.*

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

Corollary 5.5. *Let $R_m(x)$ be any polynomial of degree $< n$. Then*

$$\int_{-1}^{1} P_n(x)R_m(x)dx = 0$$

Proposition 5.6. *Let $P_n(x)$ be the n-th degree Legendre polynomial, then*

$$\langle |P_n(x)|, |P_n(x)| \rangle_I = \int_{-1}^{1} |P_n(x)|^2 = \frac{2}{2n+1}$$